Kalman-Filter revision:

More general multivariate formation:

$$\begin{array}{ll} (1) & \underset{N\times 1}{\underline{y}_t} = \underset{N\times m}{X_t}\, \underset{m\times 1}{\underline{\beta}_t} + \underset{N\times 1}{\underline{\varepsilon}_t} \qquad \underline{\varepsilon}_t \sim (\underline{0}, V_t^{(\varepsilon)}) \text{ ser. uncorr.} \end{array}$$

"measurement eqn."

$$(2) \qquad \underline{\beta}_t = T_t\, \underline{\beta}_{t-1} + Z_t\, \underline{w}_t \qquad \underline{w}_t \sim (\underline{0}, V_t^{(w)})$$

"State transition" in contemp. form
(Future form is $\underline{\beta}_t = T_{t-1}\underline{\beta}_{t-1} + R_{t-1}\underline{w}_{t-1}$.)

$\underline{y}_t$: observable variable
X_t : Observation model
$\underline{\beta}_t$: hidden (unobs) variable

include slope & intercept

Need $\underline{b}_{t|t-1} \equiv \mathbb{E}_{t-1}[\underline{\beta}_t] = \mathbb{E}[\underline{\beta}_t \mid (\underline{y}_s)_{s=1}^{t-1}]$
$R_{t|t-1} \equiv Var_{t-1}(\underline{\beta}_t) = Var(\underline{\beta}_t \mid (\underline{y}_s)_{s=1}^{t-1})$
to have a forecasting model.

Suppose $\left.\begin{array}{l} \underline{b}_{t-1} \equiv \mathbb{E}[\underline{\beta}_{t-1} \mid (\underline{y}_s)_{s=1}^{t-1}] \\ R_{t-1} \equiv Var(\underline{\beta}_{t-1} \mid (\underline{y}_s)_{s=1}^{t-1}) \end{array}\right\}$ known

$$(+)\ \left\{\begin{array}{l} \Rightarrow \underline{b}_{t|t-1} = T_t\, \underline{b}_{t-1} \\ R_{t|t-1} = T_t R_{t-1} T_t^T + Z_t V_t^{(w)} Z_t^T . \end{array}\right\} \because (2) \quad ☆$$

"prediction equations".

$$(+)\ \left\{\begin{array}{l} \tilde{y}_{t|t-1} \equiv \mathbb{E}_{t-1}[\underline{y}_t] = X_t\, \underline{b}_{t|t-1} \\ Q_{t|t-1} \equiv \underset{t-1}{Var}(\underline{y}_t) = X_t R_{t|t-1} X_t^T + V_t^{(\varepsilon)} . \end{array}\right\} \because (1)$$

↳ or just Q_t

$$\Rightarrow \begin{bmatrix} \beta_t \\ \underline{y}_t \end{bmatrix} \Big| \mathcal{F}_{t-1} \sim N\left(\begin{bmatrix} \underline{b}_{t|t-1} \\ X_t \underline{b}_{t|t-1} \end{bmatrix}, \begin{bmatrix} R_{t|t-1} & R_{t|t-1} X_t^T \\ X_t R_{t|t-1} & X_t R_{t|t-1} X_t^T + V_t^{(s)} \end{bmatrix} \right)$$

$$\left(\text{NOTE: } \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \sim N\left(\begin{bmatrix} \underline{\mu}_x \\ \underline{\mu}_y \end{bmatrix}, \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix} \right) \Rightarrow \begin{array}{l} \mathbb{E}[\underline{x} | \underline{y}] = \underline{\mu}_x + \Sigma_{xy} \Sigma_y^{-1} (\underline{y} - \underline{\mu}_y) \\ \mathrm{Var}(\underline{x} | \underline{y}) = \Sigma_x - \Sigma_{xy} \Sigma_y^{-1} \Sigma_{yx} \end{array} \right)$$

$$\Rightarrow (\dagger) \begin{cases} \underline{b}_t = \underline{b}_{t|t-1} + R_{t|t-1} X_t^T Q_{t|t-1}^{-1} (\underline{y}_t - X_t \underline{b}_{t|t-1}) \\ R_t = R_{t|t-1} - R_{t|t-1} X_t^T Q_{t|t-1}^{-1} X_t R_{t|t-1} \end{cases}$$

(Combine with ★ to get direct "updating eqn":

$$\underline{b}_{t+1|t} = T_{t+1} \underline{b}_{t|t-1} + \underbrace{T_{t-1} R_{t|t-1} X_t^T Q_{t|t-1}^{-1}}_{K_t \text{ : Kalman gain.}} (\underline{y}_t - X_t \underline{b}_{t|t-1})$$

Also get recursion for $R_{t|t-1} \to R_{t+1|t}$.
(Riccarti eqn.))

$(\dagger)$ Gives complete forecasting model using KF.

Estimate by ML.