MFE Elective, Forecasting & Financial Time Series
Part I: High-Frequency Forecasting
Appendix: the Cubic Spline Function

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This document formally defines the mathematical construction of the spline component $\varphi_{t,\tau}$.

The spline is termed a *daily* spline if the periodicity is complete over one trading day. The static daily spline assumes that the shape of intra-day periodic patterns is the same for every trading day.

The daily spline is a continuous piecewise function of time and connected at $k+1$ knots for some $k \in \mathbb{N}_{>0}$ such that $k < I$. The coordinates of the knots along the time axis are denoted by $\tau_0 < \cdots < \tau_k$, where $\tau_0 = 1$, $\tau_k = I$, and $\tau_j \in \{2, \ldots, I-1\}$ for $j = 1, \ldots, k-1$. The set of the knots is also called *mesh*. The y-coordinates (height) of the knots are denoted by $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_k)^\top$. We denote the distance between the knots along the time-axis by $h_j = \tau_j - \tau_{j-1}$ for $j = 1, \ldots k$. We begin by defining the cubic spline function $g : [\tau_0, \tau_k] \to \mathbb{R}$, which is a piecewise function of the form

$$g(\tau) = \sum_{j=1}^{k} g_j(\tau) \mathbb{1}_{\{\tau \in [\tau_{j-1}, \tau_j]\}}, \qquad \forall\, \tau \in [\tau_0, \tau_k],$$

where each function $g_j : [\tau_{j-1}, \tau_j] \to \mathbb{R}$ is a polynomial of order up to three for all $j = 1, \ldots, k$. We can set g to be continuous at each knot (τ_j, γ_j); that is, $g_j(\tau_j) = \gamma_j$ and $g_j(\tau_{j-1}) = \gamma_{j-1}$ for all $j = 1, \ldots, k$. This means we have

$$g_j(\tau_{j-1}) = g_{j-1}(\tau_{j-1}) \qquad \text{and} \qquad g_j'(\tau_{j-1}) = g_{j-1}'(\tau_{j-1}) \tag{1}$$

for $j = 2, \ldots, k$. (1) is the *continuity condition* of g. The polynomial order of each g_j means that $g_j''(\cdot)$ is a linear function on $[\tau_{j-1}, \tau_j]$ for $j = 1, \ldots, k$. This implies that

$$g_j''(\tau) = a_{j-1} + \frac{\tau - \tau_{j-1}}{h_j}(a_j - a_{j-1}) = \frac{(\tau_j - \tau)}{h_j} a_{j-1} + \frac{(\tau - \tau_{j-1})}{h_j} a_j, \tag{2}$$

for $\tau \in [\tau_{j-1}, \tau_j]$ and $j = 1, \ldots, k$, where $a_0 = g_1''(\tau_0)$ and $a_j = g_j''(\tau_j)$ for $j = 1, \ldots, k$. We call (2) the *polynomial order condition* of g.

We integrate (2) with respect to τ to find the expressions for g_j' and g_j. That is, we evaluate $g_j'(\tau) = \int g_j''(\tau) d\tau$ and $g_j(\tau) = \int \int g_j''(\tau) d\tau$ for each $j = 1, \ldots, k$, where we recover the integration constant using (1). Then we obtain

$$g_j'(\tau) \;=\; -\left[\frac{1}{2}\frac{(\tau_j - \tau)^2}{h_j} - \frac{h_j}{6}\right] a_{j-1} + \left[\frac{1}{2}\frac{(\tau - \tau_{j-1})^2}{h_j} - \frac{h_j}{6}\right] a_j, \tag{3}$$

$$g_j(\tau) \;=\; \boldsymbol{r}_j(\tau)\cdot\boldsymbol{\gamma} + \boldsymbol{s}_j(\tau)\cdot\boldsymbol{a} \tag{4}$$

for $\tau \in [\tau_{j-1}, \tau_j]$ and $j = 1, \ldots, k$, where $\boldsymbol{a} = (a_0, a_1, \ldots, a_k)^\top$, and $\boldsymbol{r}_j(\tau)$ and $\boldsymbol{s}_j(\tau)$ are the following k-dimensional vectors

$$\boldsymbol{r}_j(\tau) = \left(0, \ldots, 0, \frac{(\tau_j - \tau)}{h_j}, \frac{(\tau - \tau_{j-1})}{h_j}, 0, \ldots, 0\right)^\top,$$

$$\boldsymbol{s}_j(\tau) = \left(0, \ldots, 0, (\tau_j - \tau)\frac{(\tau_j - \tau)^2 - h_j^2}{6h_j}, (\tau - \tau_{j-1})\frac{(\tau - \tau_{j-1})^2 - h_j^2}{6h_j}, 0, \ldots, 0\right)^\top. \tag{5}$$

The non-zero elements of $\boldsymbol{r}_j(\tau)$ and $\boldsymbol{s}_j(\tau)$ are at the jth and $(j+1)$th entries.

1 Static daily spline with overnight effect

The conditions for g_j' in (1) and (3) give

$$\frac{h_j}{h_j + h_{j+1}} a_{j-1} + 2a_j + \frac{h_{j+1}}{h_j + h_{j+1}} a_{j+1} = \frac{6\gamma_{j-1}}{h_j(h_j + h_{j+1})} - \frac{6\gamma_j}{h_j h_{j+1}} + \frac{6\gamma_{j+1}}{h_{j+1}(h_j + h_{j+1})}$$

for $j = 1, \ldots, k-1$. From these, we obtained a system of $k-1$ equations with $k+1$ unknowns $a_0, \ldots, a_k$. Following Poirier (1976) we set $a_0 = a_k = 0$ (the *natural condition* for a spline). We can write this system of equations in a matrix form as $\mathbf{P}\boldsymbol{a} = \mathbf{Q}\boldsymbol{\gamma}$, where $\mathbf{P}$ and $\mathbf{Q}$ are the following square matrices of size $(k+1)$:

$$\mathbf{P} = \begin{bmatrix} 2 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \frac{h_1}{h_1+h_2} & 2 & \frac{h_2}{h_1+h_2} & 0 & \ldots & 0 & 0 \\ 0 & \frac{h_2}{h_2+h_3} & 2 & \frac{h_3}{h_2+h_3} & \cdots & 0 & 0 \\ 0 & 0 & \frac{h_3}{h_3+h_4} & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2 & \frac{h_k}{h_{k-1}+h_k} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 2 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ \frac{6}{h_1(h_1+h_2)} & -\frac{6}{h_1h_2} & \frac{6}{h_2(h_1+h_2)} & \ldots & 0 & 0 \\ 0 & \frac{6}{h_2(h_2+h_3)} & -\frac{6}{h_2h_3} & \ldots & 0 & 0 \\ 0 & 0 & \frac{6}{h_3(h_3+h_4)} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\frac{6}{h_{k-1}h_k} & \frac{6}{h_k(h_{k-1}+h_k)} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

The first and the last rows of $\mathbf{P}$ and $\mathbf{Q}$ ensure that $a_0 = a_k = 0$. For a non-singular $\mathbf{P}$, we have $\boldsymbol{a} = \mathbf{P}^{-1}\mathbf{Q}\boldsymbol{\gamma}$. Then (4) can be written as

$g_j(\tau) = \boldsymbol{w}_j(\tau) \cdot \boldsymbol{\gamma}$ for $\tau \in [\tau_{j-1}, \tau_j]$, where $\boldsymbol{w}_j(\tau)^\top = \boldsymbol{r}_j(\tau)^\top + \boldsymbol{s}_j(\tau)^\top \mathbf{P}^{-1}\mathbf{Q}$. Finally, we obtain the following expression for the daily cubic spline

$$\varphi_\tau = g(\tau) = \sum_{j=1}^{k} 1\!\!1_{\{\tau \in [\tau_{j-1}, \tau_j]\}} \boldsymbol{w}_j(\tau) \cdot \boldsymbol{\gamma}, \qquad \forall\, \tau \in [\tau_0, \tau_k]. \tag{6}$$

The elements of $\boldsymbol{\gamma}$ are the parameters of the model to be estimated. For the parameters to be identified, we impose the following zero-sum constraint on the elements of $\boldsymbol{\gamma}$:

$$\sum_{\tau \in [\tau_0, \tau_k]} \varphi_\tau = \sum_{\tau \in [\tau_0, \tau_k]} \sum_{j=1}^{k} 1\!\!1_{\{\tau \in [\tau_{j-1}, \tau_j]\}} \boldsymbol{w}_j(\tau) \cdot \boldsymbol{\gamma} = \boldsymbol{w}_* \cdot \boldsymbol{\gamma} = 0,$$

where

$$\boldsymbol{w}_* = (w_{*0}, w_{*1}, \ldots, w_{*k})^\top = \sum_{\tau \in [\tau_0, \tau_k]} \sum_{j=1}^{k} 1\!\!1_{\{\tau \in [\tau_{j-1}, \tau_j]\}} \boldsymbol{w}_j(\tau).$$

We can impose this condition by setting $\gamma_k = -\sum_{i=0}^{k-1} w_{*i}\gamma_i / w_{*k}$. Then (8) becomes

$$\varphi_\tau = \sum_{j=1}^{k} 1\!\!1_{\{\tau \in [\tau_{j-1}, \tau_j]\}} \sum_{i=0}^{k-1} \left(w_{ji}(\tau) - \frac{w_{jk}(\tau) w_{*i}}{w_{*k}} \right) \gamma_i = \sum_{j=1}^{k} 1\!\!1_{\{\tau \in [\tau_{j-1}, \tau_j]\}} \boldsymbol{z}_j(\tau) \cdot \boldsymbol{\gamma} \tag{7}$$

for $\tau \in [\tau_0, \tau_k]$. $w_{ji}(\tau)$ denotes the ith element of $\boldsymbol{w}_j(\tau)$, and the ith element of $\boldsymbol{z}_j(\tau)$ is

$$z_{ji}(\tau) = \begin{cases} w_{ji}(\tau) - w_{jk}(\tau) w_{*i} / w_{*k} & i \neq k \\ 0 & i = k \end{cases}$$

for $\tau \in [\tau_{j-1}, \tau_j]$ and each $i = 0, \ldots, k$ and $j = 1, \ldots, k$. When estimating the model, it is convenient to compute $\boldsymbol{w}_*$ using the equation $\boldsymbol{w}_*^\top = \boldsymbol{r}_*^\top + \boldsymbol{s}_*^\top \mathbf{P}^{-1}\mathbf{Q}$, where $\boldsymbol{r}_*$ and $\boldsymbol{s}_*$ are k-dimensional vectors computed using the rules of arithmetic and polynomial series as

$$\begin{aligned} \boldsymbol{r}_* &= \left(\frac{\tau_1 - \tau_0 + 1}{2}, \frac{\tau_2 - \tau_0}{2}, \ldots, \frac{\tau_{k-1} - \tau_{k-3}}{2}, \frac{\tau_k - \tau_{k-1} + 1}{2} \right)^\top, \\ \boldsymbol{s}_* &= \left(\frac{h_1 - h_1^3}{24}, \frac{\tau_2 - \tau_0 - h_2^3 - h_1^3}{24}, \ldots, \frac{\tau_{k-1} - \tau_{k-3} - h_{k-1}^3 - h_{k-2}^3}{24}, \frac{h_k - h_k^3}{24} \right)^\top. \end{aligned}$$

Note that these formulae for computing $\boldsymbol{w}_*$, $\boldsymbol{r}_*$, and $\boldsymbol{s}_*$ are different from those of Harvey and Koopman (1993) due to the removal of the periodicity condition.

2 Static weekly spline with overnight effect

The static spline becomes a static *weekly spline* if we set the periodicity to be complete over one trading week instead of one day. For this spline, recall that we redefine $\tau_0, \tau_1, \ldots, \tau_k$ as follows. We let $\widetilde{\tau}_0 < \widetilde{\tau}_1 < \cdots < \widetilde{\tau}_{k'}$ denote the coordinates along the time-axis of the *intra-day mesh*, where $k' < I$, $\widetilde{\tau}_0 = 1$, $\widetilde{\tau}_{k'} = I$, and $\widetilde{\tau}_j \in \{2, \ldots, I-1\}$ for $j = 1, \ldots, k'-1$. Then the coordinates $\tau_0, \tau_1, \ldots, \tau_k$ along the time-axis of the total mesh for the whole week is defined as $\tau_{i(k'+1)+j} = \widetilde{\tau}_j + iI$ for $i = 0, \ldots, 4$ and $j = 0, \ldots, k'$. Then $(\tau_j)_{j=0}^k$ is still an increasing sequence. The total number of knots for one whole week is $k+1 = 5(k'+1)$. The height of the knots are $\gamma_0, \gamma_1, \ldots, \gamma_{k'}$ for Monday, $\gamma_{k'+1}, \gamma_{k'+2}, \ldots, \gamma_{2(k'+1)}$ for Tuesday, and so on.

As before, there is no periodicity condition between τ_k and τ_0, so that $\gamma_k \neq \gamma_0$ is allowed and we can capture the effect of weekend news on trading patterns. Moreover, we capture the overnight effect of weeknights by relaxing the continuity and polynomial order restrictions, (1)-(2), between $\widetilde{\tau}_{k'}$ and $\widetilde{\tau}_0$ of any two successive weekdays. Thus, the procedure for computing $\boldsymbol{z}_j(\tau)$ is different from the daily spline described above. This redefines $\mathbf{P}$ and $\mathbf{Q}$ matrices as follows. For the $\mathbf{P}$ matrix, we replace the off-diagonal entries in the $i(k'+1)$th and $(i(k'+1)+1)$th rows by zeros for each $i = 1, \ldots (k+1)/(k'+1)$. For the $\mathbf{Q}$ matrix, we replace *all* entries in the $i(k'+1)$th and $(i(k'+1)+1)$th rows by zeros for each $i = 1, \ldots (k+1)/(k'+1)$. If we use five knots per day as we specified in the main text, $\mathbf{P}$ and $\mathbf{Q}$ become

$$\mathbf{P} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{h_1}{h_1+h_2} & 2 & \frac{h_2}{h_1+h_2} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{h_2}{h_2+h_3} & 2 & \frac{h_3}{h_2+h_3} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{h_3}{h_3+h_4} & 2 & \frac{h_4}{h_3+h_4} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{h_6}{h_5+h_6} & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2 & \frac{h_k}{h_{k-1}+h_k} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{6}{h_1(h_1+h_2)} & -\frac{6}{h_1h_2} & \frac{6}{h_2(h_1+h_2)} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{6}{h_2(h_2+h_3)} & -\frac{6}{h_2h_3} & \frac{6}{h_3(h_2+h_3)} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{6}{h_3(h_3+h_4)} & -\frac{6}{h_3h_4} & \frac{6}{h_4(h_3+h_4)} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{h_6(h_5+h_6)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\frac{6}{h_{k-1}h_k} & \frac{6}{h_k(h_{k-1}+h_k)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

3 Static daily spline with no overnight effect

In FX, data is collected 24 hours a day. So we assume the *periodicity condition* in this case; that is, g_1 and g_k satisfy $\gamma_0 = \gamma_k$, $g_1'(\tau_0) = g_k'(\tau_k)$, and $g_1''(\tau_0) = g_k''(\tau_k)$ so that $a_0 = a_k$. This condition is the same as Harvey and Koopman (1993) since their hourly electricity demand data is also collected 24 hours a day.

By the periodicity condition, we have $\gamma_0 = \gamma_k$ and $a_0 = a_k$ so that γ_0 and a_0 become redundant during estimation. Moreover, the conditions for g_j' in (1) and (3) give

$$\frac{h_j}{h_j + h_{j+1}} a_{j-1} + 2a_j + \frac{h_{j+1}}{h_j + h_{j+1}} a_{j+1} = \frac{6\gamma_{j-1}}{h_j(h_j + h_{j+1})} - \frac{6\gamma_j}{h_j h_{j+1}} + \frac{6\gamma_{j+1}}{h_{j+1}(h_j + h_{j+1})}$$

for $j = 2, \dots, k-1$ and

$$\frac{h_1}{h_1+h_2}a_k + 2a_1 + \frac{h_2}{h_1+h_2}a_2 = \frac{6\gamma_k}{h_1(h_1+h_2)} - \frac{6\gamma_1}{h_1 h_2} + \frac{6\gamma_2}{h_2(h_1+h_2)}, \qquad j=1,$$

$$\frac{h_k}{h_k+h_1}a_{k-1} + 2a_k + \frac{h_1}{h_k+h_1}a_1 = \frac{6\gamma_{k-1}}{h_k(h_k+h_1)} - \frac{6\gamma_k}{h_k h_1} + \frac{6\gamma_1}{h_1(h_k+h_1)} \qquad j=k.$$

From these, we obtained k equations for k "unknowns" $a_1, \dots, a_k$. Using notations $\boldsymbol{\gamma}^\dagger = (\gamma_1^\dagger, \dots, \gamma_k^\dagger)^\top = (\gamma_1, \dots, \gamma_k)^\top$ and $\boldsymbol{a}^\dagger = (a_1^\dagger, \dots, a_k^\dagger)^\top = (a_1, \dots, a_k)^\top$, we can write this system of equations in a matrix form as $\mathbf{P}\boldsymbol{a}^\dagger = \mathbf{Q}\boldsymbol{\gamma}^\dagger$, where $\mathbf{P}$ and $\mathbf{Q}$ are the following square matrices of size k:

$$\mathbf{P} = \begin{bmatrix} 2 & \frac{h_2}{h_1+h_2} & 0 & 0 & \dots & 0 & \frac{h_1}{h_1+h_2} \\ \frac{h_2}{h_2+h_3} & 2 & \frac{h_3}{h_2+h_3} & 0 & \dots & 0 & 0 \\ 0 & \frac{h_3}{h_3+h_4} & 2 & \frac{h_4}{h_3+h_4} & \cdots & 0 & 0 \\ 0 & 0 & \frac{h_4}{h_4+h_5} & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & \frac{h_k}{h_{k-1}+h_k} \\ \frac{h_1}{h_1+h_k} & 0 & 0 & 0 & \dots & \frac{h_k}{h_1+h_k} & 2 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} -\frac{6}{h_1h_2} & \frac{6}{h_2(h_1+h_2)} & 0 & \dots & 0 & \frac{6}{h_1(h_1+h_2)} \\ \frac{6}{h_2(h_2+h_3)} & -\frac{6}{h_2h_3} & \frac{6}{h_3(h_2+h_3)} & \dots & 0 & 0 \\ 0 & \frac{6}{h_3(h_3+h_4)} & -\frac{6}{h_3h_4} & \dots & 0 & 0 \\ 0 & 0 & \frac{6}{h_4(h_4+h_5)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{6}{h_{k-1}h_k} & \frac{6}{h_k(h_{k-1}+h_k)} \\ \frac{6}{h_1(h_1+h_k)} & 0 & 0 & \dots & \frac{6}{h_k(h_1+h_k)} & -\frac{6}{h_1h_k} \end{bmatrix}.$$

For a non-singular $\mathbf{P}$, we have $\boldsymbol{a}^\dagger = \mathbf{P}^{-1}\mathbf{Q}\boldsymbol{\gamma}^\dagger$. Then (4) can be written as $g_j(\tau) = \boldsymbol{w}_j(\tau) \cdot \boldsymbol{\gamma}^\dagger$ for $\tau \in [\tau_{j-1}, \tau_j]$, where $\boldsymbol{w}_j(\tau)^\top = \boldsymbol{r}_j(\tau)^\top + \boldsymbol{s}_j(\tau)^\top \mathbf{P}^{-1}\mathbf{Q}$. $\boldsymbol{r}_j(\tau)$ and $\boldsymbol{s}_j(\tau)$ are now $k \times 1$ vectors by the periodicity condition with

$$\boldsymbol{r}_1(\tau) = \left(\frac{\tau - \tau_0}{h_1}, 0, \dots, 0, \frac{\tau_1 - \tau}{h_1} \right)^\top,$$

$$\boldsymbol{s}_1(\tau) = \left((\tau - \tau_0)\frac{(\tau-\tau_0)^2 - h_1^2}{6h_1}, 0, \dots, 0, (\tau_1 - \tau)\frac{(\tau_1 - \tau)^2 - h_1^2}{6h_1} \right)^\top,$$

and $\boldsymbol{r}_j(\tau)$ and $\boldsymbol{s}_j(\tau)$ for $j = 2, \dots, k$ are as defined in (5) but the non-zero elements are shifted to $(j-1)$-th and j-th entries. Finally, we obtain the following expression for the daily cubic spline

$$\varphi_\tau = g(\tau) = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau)\cdot\boldsymbol{\gamma}^\dagger, \qquad \forall\,\tau\in[\tau_0,\tau_k]. \tag{8}$$

The elements of $\boldsymbol{\gamma}^\dagger$ are the parameters of the model to be estimated. For the parameters to be identified, we impose the following zero-sum constraint on the elements of $\boldsymbol{\gamma}^\dagger$

$$\sum_{\tau\in[\tau_0,\tau_k]} \varphi_\tau = \sum_{\tau\in[\tau_0,\tau_k]} \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau)\cdot\boldsymbol{\gamma}^\dagger = \boldsymbol{w}_*\cdot\boldsymbol{\gamma}^\dagger = 0,$$

where

$$\boldsymbol{w}_* = (w_{*0}, w_{*1}, \dots, w_{*k})^\top = \sum_{\tau\in[\tau_0,\tau_k]} \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau).$$

We can impose this condition by setting $\gamma_k = -\sum_{i=0}^{k-1} w_{*i}\gamma_i / w_{*k}$. Then (8) becomes

$$\varphi_\tau = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \sum_{i=1}^{k-1} \left(w_{ji}(\tau) - \frac{w_{jk}(\tau) w_{*i}}{w_{*k}} \right) \gamma_i^\dagger = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{z}_j(\tau)\cdot\boldsymbol{\gamma}^\dagger \tag{9}$$

for $\tau\in[\tau_0,\tau_k]$. $w_{ji}(\tau)$ denotes the ith element of $\boldsymbol{w}_j(\tau)$, and the ith element of $\boldsymbol{z}_j(\tau)$ is

$$z_{ji}(\tau) = \begin{cases} w_{ji}(\tau) - w_{jk}(\tau) w_{*i}/w_{*k} & i \neq k \\ 0 & i = k \end{cases}$$

for $\tau\in[\tau_{j-1},\tau_j]$ and $i, j = 1,\dots,k$. Thus, $\boldsymbol{z}_j : [\tau_{j-1},\tau_j]^k \to \mathbb{R}^k$ for $j = 1,\dots k$ is a k-dimensional vector of deterministic functions that conveys all information about the polynomial order, continuity, periodicity, and zero-sum conditions of the spline. (9) is the static daily spline we estimate in this chapter.

When estimating the model, it is convenient to compute $\boldsymbol{w}_*$ using the equation $\boldsymbol{w}_*^\top = \boldsymbol{r}_*^\top + \boldsymbol{s}_*^\top \mathbf{P}^{-1}\mathbf{Q}$, where $\boldsymbol{r}_*$ and $\boldsymbol{s}_*$ are k-dimensional vectors computed using the rules of arithmetic and polynomial series as

$$\boldsymbol{r}_* = \left(\frac{\tau_2-\tau_0}{2}, \dots, \frac{\tau_k-\tau_{k-2}}{2}, \frac{\tau_1-\tau_0+\tau_k-\tau_{k-1}}{2} \right)^\top,$$

$$\boldsymbol{s}_* = \left(\frac{\tau_2-\tau_0-h_2^3-h_1^3}{24}, \dots, \frac{\tau_k-\tau_{k-2}-h_k^3-h_{k-1}^3}{24}, \frac{h_1(1-h_1^2)+h_k(1-h_k^2)}{24} \right)^\top.$$

Note that these formulae for computing $\boldsymbol{w}_*$, $\boldsymbol{r}_*$, and $\boldsymbol{s}_*$ are different from the static daily spline with overnight effect.

4 Static weekly spline with no overnight effect

The way we redefine $\tau_0, \tau_1, \ldots, \tau_k$ here is slightly different to the static weekly spline with overnight effect.

$\widetilde{\tau}_0 < \widetilde{\tau}_1 < \cdots < \widetilde{\tau}_{k'}$ still denote the coordinates along the time-axis of the intra-day mesh, where $k' < I$, $\widetilde{\tau}_0 = 1$, $\widetilde{\tau}_{k'} = I$, and $\widetilde{\tau}_j \in \{2, \ldots, I-1\}$ for $j = 1, \ldots, k'-1$. Then the coordinates $\tau_0, \tau_1, \ldots, \tau_k$ along the time-axis of the total mesh for the whole week is defined as $\tau_0 = \widetilde{\tau}_0$ and $\tau_{ik'+j} = \widetilde{\tau}_j + iI$ for $i = 0, \ldots, 4$ and $j = 1, \ldots, k'$. (Note the difference here.) Then $(\tau_j)_{j=0}^{k}$ is still an increasing sequence. The total number of knots for one whole week is $k + 1 = 5k' + 1$.

The height of the knots are $\gamma_1^\dagger, \gamma_2^\dagger, \ldots, \gamma_{k'}^\dagger$ for Monday, $\gamma_{k'+1}^\dagger, \gamma_{k'+2}^\dagger, \ldots, \gamma_{2k'}^\dagger$ for Tuesday, and so on. We capture the weekend effect by allowing for $(\tau_k, \gamma_k^\dagger) \neq (\tau_0, \gamma_0^\dagger)$. The rest of the derivations that give $\boldsymbol{z}_j(\cdot)$ are the same as Section 1. This weekly spline can capture the day-of-the-week effect by allowing for

$$(\gamma_1^\dagger, \gamma_2^\dagger, \ldots, \gamma_{k'}^\dagger)^\top \neq (\gamma_{k'+1}^\dagger, \gamma_{k'+2}, \ldots, \gamma_{2k'}^\dagger)^\top \neq \cdots \neq (\gamma_{4k'+1}^\dagger, \gamma_{4k'+2}^\dagger, \ldots, \gamma_{5k'}^\dagger)^\top.$$

The restricted weekly spline is obtained by pre-multiplying $\boldsymbol{\gamma}^\dagger$ of this weekly spline by the matrix $\mathbf{S}$ as before.

REFERENCES

Harvey, A. C. and Koopman, S. J. (1993), "Forecasting Hourly Electricity Demand Using Time-Varying Splines," *Journal of the American Statistical Association*, 88, 1228–1236.

Poirier, D. (1976), *The Econometrics of Structural Change: With Special Emphasis on Spline Functions*, North-Holland.