Spline-DCS for Forecasting Trade Volume in High-Frequency Finance

Supplementary Material

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Appendix A: List of distributions

A.1 The generalized beta distribution of the second kind

The (standard) generalized beta distribution of the second kind (GB2) has the p.d.f.:

$$f(x;\nu,\xi,\zeta) = \frac{\nu x^{\nu\xi-1}(x^\nu+1)^{-\xi-\zeta}}{B(\xi,\zeta)}, \qquad x>0, \text{ and } \nu,\xi,\zeta>0$$

where $B(\cdot,\cdot)$ denotes the Beta function. GB2 becomes the Burr distribution when $\xi = 1$ and the log-logistic distribution when $\xi = \zeta = 1$. The Burr distribution is also called the Pareto Type IV distribution (Pareto IV). Log-logistic is also called Pareto III. Burr becomes Pareto II when $\nu = 1$. Burr becomes Weibull when $\zeta \to \infty$. GB2 with $\nu = 1$ and $\xi = \zeta$ is a special case of the F distribution with the degrees of freedom $\nu_1 = \nu_2 = 2\xi$. GB2 is related to the generalized gamma (GG) distribution as its limiting case.

If a non-standardized random variable Y follows the GB2 distribution, its p.d.f. $f_Y : \mathbb{R}_{>0} \to \mathbb{R}$ with the scale parameter $\alpha > 0$ is $f_Y(y;\alpha,\nu,\xi,\zeta) = f(y/\alpha;\nu,\xi,\zeta)/\alpha$ for $y > 0$. To obtain GG from GB2, we replace α by $\alpha\zeta^{1/\nu}$ and replace ζ by the tail index, $\nu\zeta$. Then GB2 becomes GG by setting $\xi = \gamma$ and letting $\nu\zeta \to \infty$. GG becomes log-normal when $\gamma \to \infty$, provided that other parameters satisfy additional conditions. See Kleiber and Kotz (2003) or Harvey and Lange (2015). For a set of i.i.d. observations $y_1, \dots, y_T$ where each follows the non-standardized GB2 distribution, the log-likelihood function of a single observation y_t can be written using the exponential link function $\alpha = \exp(\lambda)$ with the link parameter $\lambda \in \mathbb{R}$ as:

$$\log f_Y(y_t) = \log(\nu) - \nu\xi\lambda + (\nu\xi-1)\log(y_t) - \log B(\xi,\zeta) - (\xi+\zeta)\log[(y_t e^{-\lambda})^\nu + 1]. \quad \text{(A.1)}$$

The *score* u_t of the non-standardized GB2 computed at y_t is

$$u_t \equiv \frac{\partial \log f_Y(y_t)}{\partial\lambda} = \frac{\nu(\xi+\zeta)(y_t e^{-\lambda})^\nu}{(y_t e^{-\lambda})^\nu + 1} - \nu\xi = \nu(\xi+\zeta)b_t - \nu\xi \qquad \text{(A.2)}$$

where we used the notation $b_t \equiv (y_t e^{-\lambda})^\nu/((y_t e^{-\lambda})^\nu + 1)$. By the property of the GB2 distribution, we know that b_t follows the beta distribution with parameters ξ and ζ. The beta distribution characterized by the m.g.f. is

$$M_b(z;\xi,\zeta) \equiv \mathbb{E}[e^{bz}] = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\left(\frac{\xi+r}{\xi+\zeta+r}\right)\frac{z^k}{k!}\right).$$

It is easy to check that $\mathbb{E}[u_t] = 0$. $b_t(\xi,\zeta)$ is bounded between 0 and 1, which means that we have $-\nu\xi \leq u_t \leq \nu\zeta$.

A.2 The log-normal distribution

The (non-standardized) log-normal distribution has the p.d.f.:

$$f_Y(y;\alpha,\sigma) = \frac{1}{y\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\left(\frac{\log(y)-\log(\alpha)}{\sigma}\right)^2\right), \qquad y > 0, \text{ and } \alpha,\sigma > 0.$$

For a log-normally distributed random variable Y, the moments of all orders can be obtained easily using the m.g.f. of the normal distribution as $\mathbb{E}[Y^m] = \mathbb{E}[e^{m\log(Y)}]$ for all $m \in \mathbb{N}_{>0}$.

For a set of i.i.d. observations $y_1,\ldots,y_T$, where each follows the log-normal distribution, the log-likelihood function of a single observation y_t can be written using the exponential link function $\alpha = \exp(\lambda)$ as:

$$\log f_Y(y_t) = -\log(y_t) - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{1}{2}\left(\frac{\log(y_t)-\lambda}{\sigma}\right)^2.$$

The score u_t of the log-normal computed at y_t is

$$u_t \equiv \frac{\partial \log f_Y(y_t)}{\partial\lambda} = \frac{\log(y_t)-\lambda}{\sigma^2}.$$

Thus, u_t is a Gaussian random variable. We also have $\mathbb{E}[u_t] = 0$ as λ is the first moment of $\log(y_t)$.

Appendix B: The spline component

In this section, we formally explain the mathematical construction of the spline component $s_{t,\tau}$.

The spline is termed a *daily* spline if the periodicity is complete over one trading day. The static daily spline assumes that the shape of intra-day periodic patterns is the same for every trading day.

The daily spline is a continuous piecewise function of time and connected at $k+1$ knots for some $k \in \mathbb{N}_{>0}$ such that $k < I$. The coordinates of the knots along the time axis are denoted by $\tau_0 < \cdots < \tau_k$, where $\tau_0 = 1$, $\tau_k = I$, and $\tau_j \in \{2, \ldots, I-1\}$ for $j = 1, \ldots, k-1$. The set of the knots is also called *mesh*. The y-coordinates (height) of the knots are denoted by $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_k)^\top$. We denote the distance between the knots along the time-axis by $h_j = \tau_j - \tau_{j-1}$ for $j = 1, \ldots k$. We begin by defining the cubic spline function $g : [\tau_0, \tau_k] \to \mathbb{R}$, which is a piecewise function of the form

$$g(\tau) = \sum_{j=1}^{k} g_j(\tau) \mathbb{1}_{\{\tau \in [\tau_{j-1}, \tau_j]\}}, \qquad \forall\, \tau \in [\tau_0, \tau_k],$$

where each function $g_j : [\tau_{j-1}, \tau_j] \to \mathbb{R}$ is a polynomial of order up to three for all $j = 1, \ldots, k$. We can set g to be continuous at each knot (τ_j, γ_j); that is, $g_j(\tau_j) = \gamma_j$ and $g_j(\tau_{j-1}) = \gamma_{j-1}$ for all $j = 1, \ldots, k$. This means we have

$$g_j(\tau_{j-1}) = g_{j-1}(\tau_{j-1}) \qquad \text{and} \qquad g'_j(\tau_{j-1}) = g'_{j-1}(\tau_{j-1}) \tag{B.1}$$

for $j = 2, \ldots, k$. (B.1) is the *continuity condition* of g. The polynomial order of each g_j means that $g''_j(\cdot)$ is a linear function on $[\tau_{j-1}, \tau_j]$ for $j = 1, \ldots, k$. This implies that

$$g''_j(\tau) = a_{j-1} + \frac{\tau - \tau_{j-1}}{h_j}(a_j - a_{j-1}) = \frac{(\tau_j - \tau)}{h_j} a_{j-1} + \frac{(\tau - \tau_{j-1})}{h_j} a_j, \tag{B.2}$$

for $\tau \in [\tau_{j-1}, \tau_j]$ and $j = 1, \ldots, k$, where $a_0 = g''_1(\tau_0)$ and $a_j = g''_j(\tau_j)$ for $j = 1, \ldots, k$. We call (B.2) the *polynomial order condition* of g.

We integrate (B.2) with respect to τ to find the expressions for g'_j and g_j. That is, we evaluate $g'_j(\tau) = \int g''_j(\tau) d\tau$ and $g_j(\tau) = \int\int g''_j(\tau) d\tau$ for each $j = 1, \ldots, k$, where we recover the integration constant using (B.1). Then we obtain

$$g'_j(\tau) \;=\; -\left[\frac{1}{2}\frac{(\tau_j - \tau)^2}{h_j} - \frac{h_j}{6}\right] a_{j-1} + \left[\frac{1}{2}\frac{(\tau - \tau_{j-1})^2}{h_j} - \frac{h_j}{6}\right] a_j, \tag{B.3}$$

$$g_j(\tau) \;=\; \boldsymbol{r}_j(\tau) \cdot \boldsymbol{\gamma} + \boldsymbol{s}_j(\tau) \cdot \boldsymbol{a} \tag{B.4}$$

for $\tau \in [\tau_{j-1}, \tau_j]$ and $j = 1, \ldots, k$, where $\boldsymbol{a} = (a_0, a_1, \ldots, a_k)^\top$, and $\boldsymbol{r}_j(\tau)$ and $\boldsymbol{s}_j(\tau)$ are the following k-dimensional vectors

$$\boldsymbol{r}_j(\tau) = \left(0, \dots, 0, \frac{(\tau_j - \tau)}{h_j}, \frac{(\tau - \tau_{j-1})}{h_j}, 0, \dots, 0\right)^\top,$$
$$\boldsymbol{s}_j(\tau) = \left(0, \dots, 0, (\tau_j - \tau)\frac{(\tau_j - \tau)^2 - h_j^2}{6h_j}, (\tau - \tau_{j-1})\frac{(\tau - \tau_{j-1})^2 - h_j^2}{6h_j}, 0, \dots, 0\right)^\top. \quad \text{(B.5)}$$

The non-zero elements of $\boldsymbol{r}_j(\tau)$ and $\boldsymbol{s}_j(\tau)$ are at the jth and $(j+1)$th entries.

B.1 Static daily spline with overnight effect

The conditions for g_j' in (B.1) and (B.3) give

$$\frac{h_j}{h_j + h_{j+1}} a_{j-1} + 2a_j + \frac{h_{j+1}}{h_j + h_{j+1}} a_{j+1} = \frac{6\gamma_{j-1}}{h_j(h_j + h_{j+1})} - \frac{6\gamma_j}{h_j h_{j+1}} + \frac{6\gamma_{j+1}}{h_{j+1}(h_j + h_{j+1})}$$

for $j = 1, \dots, k-1$. From these, we obtained a system of $k-1$ equations with $k+1$ unknowns $a_0, \dots, a_k$. Following Poirier (1976) we set $a_0 = a_k = 0$ (the *natural condition* for a spline). We can write this system of equations in a matrix form as $\mathbf{P}\boldsymbol{a} = \mathbf{Q}\boldsymbol{\gamma}$, where $\mathbf{P}$ and $\mathbf{Q}$ are the following square matrices of size $(k+1)$:

$$\mathbf{P} = \begin{bmatrix} 2 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{h_1}{h_1+h_2} & 2 & \frac{h_2}{h_1+h_2} & 0 & \dots & 0 & 0 \\ 0 & \frac{h_2}{h_2+h_3} & 2 & \frac{h_3}{h_2+h_3} & \cdots & 0 & 0 \\ 0 & 0 & \frac{h_3}{h_3+h_4} & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & \frac{h_k}{h_{k-1}+h_k} \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{6}{h_1(h_1+h_2)} & -\frac{6}{h_1h_2} & \frac{6}{h_2(h_1+h_2)} & \dots & 0 & 0 \\ 0 & \frac{6}{h_2(h_2+h_3)} & -\frac{6}{h_2h_3} & \dots & 0 & 0 \\ 0 & 0 & \frac{6}{h_3(h_3+h_4)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{6}{h_{k-1}h_k} & \frac{6}{h_k(h_{k-1}+h_k)} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

The first and the last rows of $\mathbf{P}$ and $\mathbf{Q}$ ensure that $a_0 = a_k = 0$. For a non-singular $\mathbf{P}$, we have $\boldsymbol{a} = \mathbf{P}^{-1}\mathbf{Q}\boldsymbol{\gamma}$. Then (B.4) can be written as $g_j(\tau) = \boldsymbol{w}_j(\tau) \cdot \boldsymbol{\gamma}$ for $\tau \in [\tau_{j-1}, \tau_j]$, where $\boldsymbol{w}_j(\tau)^\top = \boldsymbol{r}_j(\tau)^\top + \boldsymbol{s}_j(\tau)^\top \mathbf{P}^{-1}\mathbf{Q}$. Finally, we obtain the following expression for the daily cubic spline

$$s_\tau = g(\tau) = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau)\cdot\boldsymbol{\gamma}, \qquad \forall\, \tau \in [\tau_0, \tau_k]. \tag{B.6}$$

The elements of $\boldsymbol{\gamma}$ are the parameters of the model to be estimated. For the parameters to be identified, we impose the following zero-sum constraint on the elements of $\boldsymbol{\gamma}$:

$$\sum_{\tau\in[\tau_0,\tau_k]} s_\tau = \sum_{\tau\in[\tau_0,\tau_k]} \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau)\cdot\boldsymbol{\gamma} = \boldsymbol{w}_*\cdot\boldsymbol{\gamma} = 0,$$

where

$$\boldsymbol{w}_* = (w_{*0}, w_{*1}, \ldots, w_{*k})^\top = \sum_{\tau\in[\tau_0,\tau_k]} \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau).$$

We can impose this condition by setting $\gamma_k = -\sum_{i=0}^{k-1} w_{*i}\gamma_i / w_{*k}$. Then (B.8) becomes

$$s_\tau = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \sum_{i=0}^{k-1} \left(w_{ji}(\tau) - \frac{w_{jk}(\tau) w_{*i}}{w_{*k}} \right) \gamma_i = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{z}_j(\tau)\cdot\boldsymbol{\gamma} \tag{B.7}$$

for $\tau \in [\tau_0, \tau_k]$. $w_{ji}(\tau)$ denotes the ith element of $\boldsymbol{w}_j(\tau)$, and the ith element of $\boldsymbol{z}_j(\tau)$ is

$$z_{ji}(\tau) = \begin{cases} w_{ji}(\tau) - w_{jk}(\tau) w_{*i}/w_{*k} & i \neq k \\ 0 & i = k \end{cases}$$

for $\tau \in [\tau_{j-1}, \tau_j]$ and each $i = 0, \ldots, k$ and $j = 1, \ldots, k$. When estimating the model, it is convenient to compute $\boldsymbol{w}_*$ using the equation $\boldsymbol{w}_*^\top = \boldsymbol{r}_*^\top + \boldsymbol{s}_*^\top \mathbf{P}^{-1}\mathbf{Q}$, where $\boldsymbol{r}_*$ and $\boldsymbol{s}_*$ are k-dimensional vectors computed using the rules of arithmetic and polynomial series as

$$\begin{aligned} \boldsymbol{r}_* &= \left(\frac{\tau_1 - \tau_0 + 1}{2}, \frac{\tau_2 - \tau_0}{2}, \ldots, \frac{\tau_{k-1} - \tau_{k-3}}{2}, \frac{\tau_k - \tau_{k-1} + 1}{2} \right)^\top, \\ \boldsymbol{s}_* &= \left(\frac{h_1 - h_1^3}{24}, \frac{\tau_2 - \tau_0 - h_2^3 - h_1^3}{24}, \ldots, \frac{\tau_{k-1} - \tau_{k-3} - h_{k-1}^3 - h_{k-2}^3}{24}, \frac{h_k - h_k^3}{24} \right)^\top. \end{aligned}$$

Note that these formulae for computing $\boldsymbol{w}_*$, $\boldsymbol{r}_*$, and $\boldsymbol{s}_*$ are different from those of Harvey and Koopman (1993) due to the removal of the periodicity condition.

B.2 Static daily spline with no overnight effect

Since the FX data is collected 24 hours a day, we assume the *periodicity condition* in this case; that is, g_1 and g_k satisfy $\gamma_0 = \gamma_k$, $g_1'(\tau_0) = g_k'(\tau_k)$, and $g_1''(\tau_0) = g_k''(\tau_k)$ so that $a_0 = a_k$. This condition is the same as Harvey and Koopman (1993) since their hourly electricity demand data is also collected 24 hours a day.

By the periodicity condition, we have $\gamma_0 = \gamma_k$ and $a_0 = a_k$ so that γ_0 and a_0 become redundant during estimation. Moreover, the conditions for g_j' in (B.1) and (B.3) give

$$\frac{h_j}{h_j + h_{j+1}} a_{j-1} + 2a_j + \frac{h_{j+1}}{h_j + h_{j+1}} a_{j+1} = \frac{6\gamma_{j-1}}{h_j(h_j + h_{j+1})} - \frac{6\gamma_j}{h_j h_{j+1}} + \frac{6\gamma_{j+1}}{h_{j+1}(h_j + h_{j+1})}$$

for $j = 2, \dots, k-1$ and

$$\frac{h_1}{h_1 + h_2} a_k + 2a_1 + \frac{h_2}{h_1 + h_2} a_2 = \frac{6\gamma_k}{h_1(h_1 + h_2)} - \frac{6\gamma_1}{h_1 h_2} + \frac{6\gamma_2}{h_2(h_1 + h_2)}, \qquad j = 1,$$

$$\frac{h_k}{h_k + h_1} a_{k-1} + 2a_k + \frac{h_1}{h_k + h_1} a_1 = \frac{6\gamma_{k-1}}{h_k(h_k + h_1)} - \frac{6\gamma_k}{h_k h_1} + \frac{6\gamma_1}{h_1(h_k + h_1)} \qquad j = k.$$

From these, we obtained k equations for k "unknowns" $a_1, \dots, a_k$. Using notations $\boldsymbol{\gamma}^\dagger = (\gamma_1^\dagger, \dots, \gamma_k^\dagger)^\top = (\gamma_1, \dots, \gamma_k)^\top$ and $\boldsymbol{a}^\dagger = (a_1^\dagger, \dots, a_k^\dagger)^\top = (a_1, \dots, a_k)^\top$, we can write this system of equations in a matrix form as $\mathbf{P}\boldsymbol{a}^\dagger = \mathbf{Q}\boldsymbol{\gamma}^\dagger$, where $\mathbf{P}$ and $\mathbf{Q}$ are the following square matrices of size k:

$$\mathbf{P} = \begin{bmatrix} 2 & \frac{h_2}{h_1+h_2} & 0 & 0 & \dots & 0 & \frac{h_1}{h_1+h_2} \\ \frac{h_2}{h_2+h_3} & 2 & \frac{h_3}{h_2+h_3} & 0 & \dots & 0 & 0 \\ 0 & \frac{h_3}{h_3+h_4} & 2 & \frac{h_4}{h_3+h_4} & \dots & 0 & 0 \\ 0 & 0 & \frac{h_4}{h_4+h_5} & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & \frac{h_k}{h_{k-1}+h_k} \\ \frac{h_1}{h_1+h_k} & 0 & 0 & 0 & \dots & \frac{h_k}{h_1+h_k} & 2 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} -\frac{6}{h_1h_2} & \frac{6}{h_2(h_1+h_2)} & 0 & \dots & 0 & \frac{6}{h_1(h_1+h_2)} \\ \frac{6}{h_2(h_2+h_3)} & -\frac{6}{h_2h_3} & \frac{6}{h_3(h_2+h_3)} & \dots & 0 & 0 \\ 0 & \frac{6}{h_3(h_3+h_4)} & -\frac{6}{h_3h_4} & \dots & 0 & 0 \\ 0 & 0 & \frac{6}{h_4(h_4+h_5)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{6}{h_{k-1}h_k} & \frac{6}{h_k(h_{k-1}+h_k)} \\ \frac{6}{h_1(h_1+h_k)} & 0 & 0 & \dots & \frac{6}{h_k(h_1+h_k)} & -\frac{6}{h_1h_k} \end{bmatrix}.$$

For a non-singular $\mathbf{P}$, we have $\boldsymbol{a}^\dagger = \mathbf{P}^{-1}\mathbf{Q}\boldsymbol{\gamma}^\dagger$. Then (B.4) can be written as $g_j(\tau) = \boldsymbol{w}_j(\tau) \cdot \boldsymbol{\gamma}^\dagger$ for $\tau \in [\tau_{j-1}, \tau_j]$, where $\boldsymbol{w}_j(\tau)^\top = \boldsymbol{r}_j(\tau)^\top + \boldsymbol{s}_j(\tau)^\top \mathbf{P}^{-1}\mathbf{Q}$. $\boldsymbol{r}_j(\tau)$ and $\boldsymbol{s}_j(\tau)$ are now $k \times 1$ vectors by the periodicity condition with

$$\begin{aligned}\boldsymbol{r}_1(\tau) &= \left(\frac{\tau-\tau_0}{h_1}, 0, \ldots, 0, \frac{\tau_1-\tau}{h_1}\right)^\top, \\ \boldsymbol{s}_1(\tau) &= \left((\tau-\tau_0)\frac{(\tau-\tau_0)^2-h_1^2}{6h_1}, 0, \ldots, 0, (\tau_1-\tau)\frac{(\tau_1-\tau)^2-h_1^2}{6h_1}\right)^\top,\end{aligned}$$

and $\boldsymbol{r}_j(\tau)$ and $\boldsymbol{s}_j(\tau)$ for $j = 2, \ldots, k$ are as defined in (B.5), but the non-zero elements are shifted to $(j-1)$-th and j-th entries. Finally, we obtain the following expression for the daily cubic spline

$$s_\tau = g(\tau) = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau)\cdot\boldsymbol{\gamma}^\dagger, \qquad \forall\, \tau \in [\tau_0, \tau_k]. \tag{B.8}$$

The elements of $\boldsymbol{\gamma}^\dagger$ are the parameters of the model to be estimated. For the parameters to be identified, we impose the following zero-sum constraint on the elements of $\boldsymbol{\gamma}^\dagger$

$$\sum_{\tau\in[\tau_0,\tau_k]} s_\tau = \sum_{\tau\in[\tau_0,\tau_k]} \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau)\cdot\boldsymbol{\gamma}^\dagger = \boldsymbol{w}_*\cdot\boldsymbol{\gamma}^\dagger = 0,$$

where

$$\boldsymbol{w}_* = (w_{*0}, w_{*1}, \ldots, w_{*k})^\top = \sum_{\tau\in[\tau_0,\tau_k]} \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{w}_j(\tau).$$

We can impose this condition by setting $\gamma_k = -\sum_{i=0}^{k-1} w_{*i}\gamma_i / w_{*k}$. Then (B.8) becomes

$$s_\tau = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \sum_{i=1}^{k-1} \left(w_{ji}(\tau) - \frac{w_{jk}(\tau)w_{*i}}{w_{*k}}\right)\gamma_i^\dagger = \sum_{j=1}^{k} \mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}} \boldsymbol{z}_j(\tau)\cdot\boldsymbol{\gamma}^\dagger \tag{B.9}$$

for $\tau \in [\tau_0, \tau_k]$. $w_{ji}(\tau)$ denotes the ith element of $\boldsymbol{w}_j(\tau)$, and the ith element of $\boldsymbol{z}_j(\tau)$ is

$$z_{ji}(\tau) = \begin{cases} w_{ji}(\tau) - w_{jk}(\tau)w_{*i}/w_{*k} & i \neq k \\ 0 & i = k \end{cases}$$

for $\tau \in [\tau_{j-1}, \tau_j]$ and $i, j = 1, \ldots, k$. Thus, $\boldsymbol{z}_j : [\tau_{j-1}, \tau_j]^k \to \mathbb{R}^k$ for $j = 1, \ldots k$ is a k-dimensional vector of deterministic functions that conveys all information about the polynomial order, continuity, periodicity, and zero-sum conditions of the spline. (B.9) is the static daily spline we estimate in this chapter.

When estimating the model, it is convenient to compute $\boldsymbol{w}_*$ using the equation $\boldsymbol{w}_*^\top = \boldsymbol{r}_*^\top + \boldsymbol{s}_*^\top \mathbf{P}^{-1}\mathbf{Q}$, where $\boldsymbol{r}_*$ and $\boldsymbol{s}_*$ are k-dimensional vectors computed using the rules of arithmetic and polynomial series as

$$\boldsymbol{r}_* = \left(\frac{\tau_2-\tau_0}{2},\ldots,\frac{\tau_k-\tau_{k-2}}{2},\frac{\tau_1-\tau_0+\tau_k-\tau_{k-1}}{2}\right)^\top,$$
$$\boldsymbol{s}_* = \left(\frac{\tau_2-\tau_0-h_2^3-h_1^3}{24},\ldots,\frac{\tau_k-\tau_{k-2}-h_k^3-h_{k-1}^3}{24},\frac{h_1(1-h_1^2)+h_k(1-h_k^2)}{24}\right)^\top.$$

Note that these formulae for computing $\boldsymbol{w}_*$, $\boldsymbol{r}_*$, and $\boldsymbol{s}_*$ are different from the static daily spline with overnight effect.

Appendix C: Asymptotic behavior of MLE

We simulate the asymptotic distribution of MLE in the following simple version of spline-DCS:

$$y_{t,\tau} = \varepsilon_{t,\tau}\exp(\lambda_{t,\tau}), \quad \varepsilon_{t,\tau} \sim \text{i.i.d. GB2}(\nu,\xi,\zeta), \quad \lambda_{t,\tau} = \omega + \mu_{t,\tau} + \eta_{t,\tau} + s_{t,\tau},$$
$$\mu_{t,\tau} = \mu_{t,\tau-1} + \kappa_\mu u_{t,\tau-1}, \quad \eta_{t,\tau} = \phi_1\eta_{t,\tau-1} + \kappa_\eta u_{t,\tau-1}.$$

In this simulation, we assume that there are $I = 100$ bins per day and up to $T = 50$ sample days, giving up to $I \times T = 5,000$ samples in total. $s_{t,\tau}$ is a static daily spline with no overnight effect (as in FX) and 3 distinct knots per day at $\tau_0 = 1$, $\tau_1 = \lfloor I/3 \rfloor = 33$, and $\tau_2 = 2\lfloor I/3 \rfloor = 66$. We simulate this model $K = 1,000$ times and compute the estimator at each simulation.

The simulation results in Table 1 suggest that biases and the size of errors generally decrease as sample size increases, suggesting consistency.[1] The 95% coverage probabilities seem to validate standard statistical inference for model selection using this estimator at sample size as large as $T \times I = 5,000$. The Kolmogorov-Smirnov (KS) statistics testing the null of Gausianity of MLE at $T \times I = 5,000$ are outside the rejection region at the 5% level for all parameters except κ_μ. But this result for κ_μ appears inconsequential given the 95% coverage probability, and may improve by changing optimization tolerance or increasing the size of simulation from $K = 1,000$.

[1] The median bias and absolute deviation quantities are more reliable than the mean bias or squared error quantities, as we do not know the existence of these moments in small sample.

	Mean Bias				Mean Sq. Error				Median Bias			
T×I	500	1000	2000	5000	500	1000	2000	5000	500	1000	2000	5000
κ_μ	-0.02	-0.01	0.00	0.00	0.0008	0.0002	0.0001	0.0000	-0.021	-0.005	-0.001	0.000
$\phi_1^{(1)}$	-0.03	-0.01	-0.01	0.00	0.0060	0.0026	0.0008	0.0002	-0.008	-0.002	-0.002	0.000
$\kappa_\eta^{(1)}$	0.00	0.00	0.00	0.00	0.0010	0.0003	0.0001	0.0000	0.0007	0.0001	0.0004	-0.0001
γ_0	0.00	0.01	0.00	0.00	0.0365	0.0259	0.0175	0.0044	-0.007	0.008	-0.005	-0.002
γ_1	0.00	0.00	0.00	0.00	0.0070	0.0045	0.0029	0.0012	-0.002	-0.003	0.001	-0.002
γ_2	0.01	0.00	0.00	0.00	0.0066	0.0041	0.0022	0.0010	0.002	0.001	0.001	0.002
ν	0.01	0.01	-0.01	0.00	0.0438	0.0379	0.0372	0.0306	-0.003	-0.006	-0.027	-0.008
ξ	0.03	0.03	0.03	0.02	0.0335	0.0291	0.0248	0.0176	0.021	0.011	0.023	0.006
ζ	0.04	0.02	0.03	0.02	0.0367	0.0263	0.0246	0.0174	0.024	0.011	0.020	0.009
ω	0.00	-0.01	0.02	0.00	0.0472	0.0462	0.0578	0.0505	-0.008	-0.011	0.036	-0.006

	Med. Abs. Dev.				95% Cov. Prob.				KS Test (p-val)	True Values
T×I	500	1000	2000	5000	500	1000	2000	5000	5000	
κ_μ	0.023	0.008	0.004	0.002	0.74	0.81	0.85	0.94	0.003	0.01
ϕ_1	0.024	0.019	0.015	0.009	0.89	0.90	0.91	0.93	0.410	0.95
κ_η	0.016	0.011	0.007	0.004	0.92	0.96	0.96	0.98	0.143	0.05
γ_0	0.141	0.109	0.090	0.041	0.94	0.86	0.82	0.92	0.062	1.2
γ_1	0.051	0.047	0.035	0.023	0.99	0.98	0.96	0.94	0.833	-0.4
γ_2	0.053	0.040	0.032	0.022	0.99	0.97	0.98	0.97	0.138	-0.2
ν	0.150	0.138	0.146	0.131	1.00	1.00	1.00	0.96	0.640	2
ξ	0.122	0.114	0.108	0.091	1.00	1.00	0.99	0.95	0.075	1
ζ	0.133	0.112	0.101	0.093	1.00	1.00	0.99	0.95	0.188	1
ω	0.151	0.157	0.204	0.176	0.99	0.98	0.97	0.97	0.056	9

Table 1. Selected statistics from the simulated asymptotic distribution of MLE in a simple spline-DCS model. The sample length of $I \times T = 5,000$ is simulated $K = 1,000$ times and the MLE is computed at each simulation to simulate its asymptotic distribution. The KS statistics test the null that $\sqrt{I \times T}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \sim N(0, \mathcal{I}(\boldsymbol{\theta}_0)^{-1})$ at $I \times T = 5,000$, where $\mathcal{I}(\boldsymbol{\theta}_0)$ is the information quantity and $\boldsymbol{\theta}_0$ is the vector of true parameter values. The in-sample computation of the information quantity is as outlined in Appendix D.

Appendix D: Derivatives for standard errors

Let $\boldsymbol{\vartheta}$ denote the vector of all of the constant parameters of spline-DCS including the distribution parameters in $\boldsymbol{\theta}$. Denoting the single log-likelihood of an observation, $y_{t,\tau}$, as $\log f_Y(y_{t,\tau}; \boldsymbol{\vartheta})$,[2] we compute the standard error of MLE for the i-th element of $\boldsymbol{\vartheta}$ (denoted by $\widehat{\boldsymbol{\vartheta}}_i$) using the outer-product of the first-derivative of the joint log-likelihood as:

$$\text{S.E.}(\widehat{\boldsymbol{\vartheta}}_{\text{i}}) = \sqrt{\left(\sum_{(t,\tau)\in\Psi_{T,I}} \frac{\partial \log f_Y(y_{t,\tau}; \widehat{\boldsymbol{\vartheta}})}{\partial \boldsymbol{\vartheta}} \frac{\partial \log f_Y(y_{t,\tau}; \widehat{\boldsymbol{\vartheta}})}{\partial \boldsymbol{\vartheta}^\top} \right)^{-1}_{ii}},$$

where $\cdot_{ii}$ for $i = 1, \dots, \dim(\boldsymbol{\vartheta})$ denotes the i-th diagonal element.

If F^* is GB2, $\boldsymbol{\vartheta}$ includes $\boldsymbol{\theta} = (\boldsymbol{\theta}^{*\top}, p)^\top$ with $\boldsymbol{\theta}^* = (\nu, \xi, \zeta)^\top$, as well as the parameters

[2]The relationship between f_Y and f are given in Appendix A.

of $\lambda_{t,\tau}$. Then we have the following derivatives for the parameters in $\boldsymbol{\theta}$:

$$\begin{aligned}
\frac{\partial \log f_Y(y_{t,\tau};\boldsymbol{\vartheta})}{\partial \nu} &= \frac{1}{\nu} + \xi \log(y_{t,\tau}e^{-\lambda_{t,\tau}}) + u_{t,\tau}\frac{\partial \lambda_{t,\tau}}{\partial \nu} - (\xi+\zeta)b_{t,\tau}\log(y_{t,\tau}e^{-\lambda_{t,\tau}}),\\
\frac{\partial \log f_Y(y_{t,\tau};\boldsymbol{\vartheta})}{\partial \xi} &= \log(b_{t,\tau}) - \frac{1}{B(\xi,\zeta)}\frac{\partial B(\xi,\zeta)}{\partial \xi} + u_{t,\tau}\frac{\partial \lambda_{t,\tau}}{\partial \xi},\\
\frac{\partial \log f_Y(y_{t,\tau};\boldsymbol{\vartheta})}{\partial \zeta} &= -\frac{1}{B(\xi,\zeta)}\frac{\partial B(\xi,\zeta)}{\partial \zeta} + \log(1-b_{t,\tau}) + u_{t,\tau}\frac{\partial \lambda_{t,\tau}}{\partial \zeta},\\
\frac{\partial \log f_Y(y_{t,\tau};\boldsymbol{\vartheta})}{\partial p} &= \mathbb{1}_{\{y_{t,\tau}=0\}}/p - \mathbb{1}_{\{y_{t,\tau}>0\}}/(1-p),
\end{aligned}$$

where $u_{t,\tau} \equiv \nu(\xi+\zeta)b_{t,\tau} - \nu\xi$ is the score of GB2 and $b_{t,\tau} \equiv 1/(1+(y_{t,\tau}\exp(-\lambda_{t,\tau}))^{-\nu})$. (See (A.1) and (A.2) in Appendix A.1). We also have $\partial B(\xi,\zeta)/\partial\xi = B(\xi,\zeta)(\psi(\xi)-\psi(\xi+\zeta))$ with $\partial B(\xi,\zeta)/\partial\zeta = \partial B(\zeta,\xi)/\partial\zeta$ by the symmetry of the Beta function. $\psi(\cdot)$ is the digamma function. For other parameters of $\boldsymbol{\vartheta}$ (denoted by $\boldsymbol{\vartheta}_{-\boldsymbol{\theta}}$), we have

$$\frac{\partial \log f_Y(y_{t,\tau};\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}_{-\boldsymbol{\theta},i}} = u_{t,\tau}\frac{\partial \lambda_{t,\tau}}{\partial \boldsymbol{\vartheta}_{-\boldsymbol{\theta},i}},$$

where $\boldsymbol{\vartheta}_{-\boldsymbol{\theta},i}$ is the i-th element of $\boldsymbol{\vartheta}_{-\boldsymbol{\theta}}$. Denoting the i-th element of $\boldsymbol{\vartheta}$ by $\boldsymbol{\vartheta}_i$, the derivatives of $\lambda_{t,\tau}$ are given by the following recursions:

$$\begin{aligned}
\frac{\partial \lambda_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} &= \mathbb{1}_{\{\boldsymbol{\vartheta}_i=\omega\}} + \frac{\partial \mu_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} + \frac{\partial \eta_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} + \frac{\partial s_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} + \frac{\partial e_{t,\tau}}{\partial \boldsymbol{\vartheta}_i}\\
\frac{\partial \mu_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} &= \frac{\partial \mu_{t,\tau-1}}{\partial \boldsymbol{\vartheta}_i} + u_{t,\tau-1}\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\kappa_\mu\}} + \kappa_\mu\frac{\partial u_{t,\tau-1}}{\partial \boldsymbol{\vartheta}_i}\\
\frac{\partial \eta_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} &= \frac{\partial \eta^{(1)}_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} + \frac{\partial \eta^{(2)}_{t,\tau}}{\partial \boldsymbol{\vartheta}_i}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \eta^{(j)}_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} &= \eta^{(j)}_{t,\tau-1}\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\phi^{(j)}_1\}} + \eta^{(j)}_{t,\tau-2}\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\phi^{(1)}_2\}} + u_{t,\tau-1}\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\kappa^{(j)}_\eta\}}\\
&\quad + \phi^{(j)}_1\frac{\partial \eta^{(j)}_{t,\tau-1}}{\partial \boldsymbol{\vartheta}_i} + \phi^{(j)}_2\frac{\partial \eta^{(j)}_{t,\tau-2}}{\partial \boldsymbol{\vartheta}_i} + \kappa^{(j)}_\eta\frac{\partial u_{t,\tau-1}}{\partial \boldsymbol{\vartheta}_i}\\
&\quad + \mathrm{sign}(-r_{t,\tau})(u_{t,\tau-1}+\nu\xi)\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\kappa^{(j)}_{\eta,a}\}}\\
&\quad + \kappa^{(j)}_{\eta,a}\mathrm{sign}(-r_{t,\tau})\left(\frac{\partial u_{t,\tau-1}}{\partial \boldsymbol{\vartheta}_i} + \xi\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\nu\}} + \nu\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\xi\}}\right), \quad j=1,2,\\
\frac{\partial e_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} &= e_{t,\tau-1}\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\phi_e\}} + \phi_e\frac{\partial e_{t,\tau-1}}{\partial \boldsymbol{\vartheta}_i} + d_{t,\tau,m}\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\kappa_{e,m}\}}, \quad m=1,\ldots,\dim(\boldsymbol{d}_{t,\tau}),
\end{aligned}$$

where $\kappa_{e,m}$ is m-th element of $\boldsymbol{\kappa}_e$. As for the spline component, we have

$$\frac{\partial s_\tau}{\partial \boldsymbol{\vartheta}_i} = \sum_{j=1}^{k}\mathbb{1}_{\{\tau\in[\tau_{j-1},\tau_j]\}}\boldsymbol{z}_{j,l}(\tau)\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\gamma_l\}}.$$

where $l=0,\ldots,k-1$ and k is the number of knots. Given our assumption about

$\mathcal{F}_{1,1}$, these recursions are assumed to begin from zero. Finally, for the score variable, we have

$$\frac{\partial u_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} = ((\xi+\zeta)b_{t,\tau} - \xi)\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\nu\}} + \nu(b_{t,\tau}-1)\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\xi\}} + \nu b_{t,\tau}\mathbb{1}_{\{\boldsymbol{\vartheta}_i=\zeta\}} + \nu(\xi+\zeta)\frac{\partial b_{t,\tau}}{\partial \boldsymbol{\vartheta}_i}$$

$$\frac{\partial b_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} = \begin{cases} -b_{t,\tau}(1-b_{t,\tau})\log(y_{t,\tau}e^{-\lambda_{t,\tau}})(y_{t,\tau}e^{-\lambda_{t,\tau}})\frac{\partial \lambda_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} & \text{if } \boldsymbol{\vartheta}_i = \nu, \\ -\nu b_{t,\tau}(1-b_{t,\tau})\frac{\partial \lambda_{t,\tau}}{\partial \boldsymbol{\vartheta}_i} & \text{otherwise.} \end{cases}$$

Appendix E: Additional in-sample results for spline-DCS

Pair	USDJPY		EURUSD					
Window	11		13		6		14	
	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error
ν	1.576	0.017	1.702	0.016	1.973	0.021	1.844	0.021
ξ	1.459	0.052	1.605	0.063	1.089	0.037	1.427	0.049
ζ	1.982	0.078	1.866	0.072	1.538	0.062	2.119	0.088
ω	4.414	0.088	4.184	0.084	3.366	0.072	5.067	0.073
κ_μ	0.031	0.006	0.013	0.004	0.010	0.005	0.004	0.003
$\phi_1^{(1)}$	0.439	0.091	0.581	0.024	0.486	0.011	0.550	0.017
$\phi_2^{(1)}$	0.258	0.091	0.328	0.024	0.495	0.011	0.384	0.017
$\kappa_\eta^{(1)}$	0.087	0.012	0.091	0.009	0.062	0.008	0.079	0.007
$\phi_1^{(2)}$	0.522	0.107	0.608	0.080	0.638	0.073	0.526	0.079
$\kappa_\eta^{(2)}$	0.094	0.013	0.092	0.010	0.094	0.011	0.086	0.009
ϕ_e	0.582	0.089	0.819	0.036	0.477	0.135	0.755	0.066
p	0.015	0.003	0.009	0.002	0.022	0.003	0.008	0.002
$\kappa_{e,1}$	2.854	0.021	2.527	0.021	0.575	0.089	2.632	0.021
$\kappa_{e,2}$	1.130	0.161	0.825	0.139	0.214	0.021	0.569	0.085
$\kappa_{e,3}$	NaN	NaN	0.677	0.728	0.025	2.049	0.570	0.827

Table 2. The estimated parameter values for spline-DCS for selected in-sample windows. The standard errors are computed analytically as described in Section D.

Appendix F: Fourier-MEM Model

We adhere to the notations of Brownlees et al. (2011). The version of their model we test in this paper is

$$\begin{aligned}
y_{t,\tau} &= \eta_t \phi_\tau \mu_{t,\tau} e^*_{t,\tau} \varepsilon_{t,\tau}, \quad \varepsilon_{t,\tau} \text{ i.i.d.} \sim (1, \sigma^2), \\
\eta_t &= \alpha_0^{(\eta)} + \beta_1^{(\eta)} \eta_{t-1} + \alpha_1^{(\eta)} y_{t-1}^{(\eta)} \\
\mu_{t,\tau} &= \alpha_0^{(\mu)} + \beta_1^{(\mu)} \mu_{t,\tau-1} + \alpha_1^{(\mu)} y_{t,\tau-1}^{(\mu)} + \alpha_2^{(\mu)} y_{t,\tau-2}^{(\mu)} \\
\phi_{\tau+1} &= \exp\left\{ \sum_{k=1}^{\lfloor I/2 \rfloor} \left[\delta_{1k} \cos\left(\frac{2\pi k\tau}{I}\right) + \delta_{2k} \sin\left(\frac{2\pi k\tau}{I}\right) \right] \right\}, \\
e^*_{t,\tau} &= \exp(e_{t,\tau}), \quad e_{t,\tau} = \beta_1^{(e)} e_{t,\tau-1} + \boldsymbol{\alpha}_1^{(e)\top} \boldsymbol{d}_{t,\tau}.
\end{aligned}$$

η_t is the daily component, $\mu_{t,\tau}$ is the intra-day non-periodic component, and ϕ_τ is the intra-day periodic component, which approximate periodic patterns using the Fourier series. A new component we add to their model is $e^*_{t,\tau}$, which captures the effect of anticipated macroeconomic announcements. $\boldsymbol{d}_{t,\tau}$ is as defined in (1) of the main document, and $\dim(\boldsymbol{\alpha}_1^{(e)}) = m$. $y_t^{(\eta)}$ and $y_{t,\tau}^{(\mu)}$ are the *standardized* daily volume and intra-daily volume, respectively, which are defined as

$$y_t^{(\eta)} = \eta_t I^{-1} \sum_{\tau=1}^{I} \varepsilon_{t,\tau}, \quad \text{and} \quad y_{t,\tau}^{(\mu)} = \mu_{t,\tau} \varepsilon_{t,\tau}.$$

These make the model analogous to the GARCH model in the return volatility literature.[3] There are no asymmetry terms due to the reasons we discussed in Section 3.3.1 of the main document.

The interpretations and identification conditions of the parameters are as laid out by the authors. To give a refresher, we mention some of them here. $\mu_{t,\tau}$ is stationary if $|\beta_1^{(\mu)} + \alpha_1^{(\mu)} + \alpha_2^{(\mu)}| < 1$. η_t is stationary if $|\beta_1^{(\eta)} + \alpha_1^{(\eta)}| < 1$. For the identification of the parameters, we assume $\mathbb{E}[\mu_{t,\tau}] = 1$, giving the constraint $\alpha_0^{(\mu)} = 1 - (\beta_1^{(\mu)} + \alpha_1^{(\mu)} + \alpha_2^{(\mu)})$. The initial conditions are $\eta_1 = x_1^{(\eta)} = \sum_{t=1}^{5} \sum_{\tau=1}^{I} y_{t,\tau}/5$ (i.e. the daily average volume over

[3] η_t and $\mu_{t,\tau}$ are GARCH-like filters because

$$\mathbb{E}[y_{t,\tau}^{(\mu)} | \mathcal{F}_{t,\tau-1}] = \mu_{t,\tau}, \qquad \text{Var}[y_{t,\tau}^{(\mu)} | \mathcal{F}_{t,\tau-1}] = \mu_{t,\tau}^2 \sigma^2,$$

and likewise for $y_t^{(\eta)}$ (with the normalization factor $1/I$ for the variance). The conditional moment of volume in the τ-th bin on the t-th observation day is $\mathbb{E}[y_{t,\tau} | \mathcal{F}_{t,\tau-1}] = \eta_t \phi_\tau \mu_{t,\tau}$.

the first observation week excluding the weekend period) and $\mu_{1,1} = y^{(\mu)}_{1,1} = \mathbb{E}[\mu_{t,\tau}] = 1$. The continuity condition of data is $\mu_{t,0} = \mu_{t-1,I}$ and $y^{(\mu)}_{t,0} = y^{(\mu)}_{t-1,I}$. A sufficient condition to ensure that volume is positive is that the parameters of η_t and $\mu_{t,\tau}$ are positive. Taking the exponential of the Fourier series ensures that ϕ_τ is positive. $\delta_{2\lfloor I/2 \rfloor} = 0$ if I is even. ϕ_τ assumes that the pattern of periodicity is the same every day. In our case, there is no overnight dummies of the type used by Brownlees et al. (2011) in $\mu_{t,\tau}$ for the reasons described in Section 3.1 of the main document, as we are modeling the FX data.

Brownlees et al. (2011) apply the model to forecasting the volume turnover of three Exchange Traded Funds that replicate the movements of U.S. stock indices, SPDR S&P 500, Diamonds, and PowerShares QQQ, between 2002 and 2006. The highest sampling frequency they consider is 15 minutes. The authors find that their intra-day volume data tend to cluster and that there is a strong serial-correlation in daily average volume. They also find that their data displays diurnal U-shaped patterns, with low volume around noon and high volume at the beginning and the end of trading day. This is similar to the periodic patterns we found in our equity data, but different to the patterns we found in our FX volume data. Our in-sample estimation results indicated that the model with the second lag-term in $\mu_{t,\tau}$ generally performed well in capturing the volume dynamics, which is consistent with Brownlees et al. (2011).

Appendix G: Fourier-MEM estimation

Let $\boldsymbol{\varphi}$ denote the vector of all parameters of Fourier-MEM. Brownlees et al. (2011) show that the GMM estimator, $\widehat{\boldsymbol{\varphi}}_{IT}$, of $\boldsymbol{\varphi}$ solves the MM equation

$$\frac{1}{IT}\sum_{t=1}^{T}\sum_{\tau=1}^{I} \boldsymbol{a}_{t,\tau} u_{t,\tau} = \mathbf{0},$$

where

$$\begin{aligned}
\boldsymbol{a}_{t,\tau} &= \eta_t^{-1}\nabla_{\boldsymbol{\varphi}}\eta_t + \mu_{t,\tau}^{-1}\nabla_{\boldsymbol{\varphi}}\mu_{t,\tau} + \phi_\tau^{-1}\nabla_{\boldsymbol{\varphi}}\phi_\tau + e^{*-1}_{t,\tau}\nabla_{\boldsymbol{\varphi}}e^*_{t,\tau},\\
u_{t,\tau} &= y_{t,\tau}/(\eta_t\phi_\tau\mu_{t,\tau}e^*_{t,\tau}) - 1.
\end{aligned}$$

Under certain regularity conditions, $\widehat{\boldsymbol{\varphi}}_{IT}$ is asymptotically normal. The asymptotic covariance matrix is consistently estimated by

$$\widehat{\text{Avar}}(\widehat{\boldsymbol{\varphi}}_{IT}) = \frac{1}{IT} \sum_{t=1}^{T} \sum_{\tau=1}^{I} \widehat{u}_{t,\tau}^{2} \left[\sum_{t=1}^{T} \sum_{\tau=1}^{I} \boldsymbol{a}_{t,\tau} \boldsymbol{a}_{t,\tau}^{\top} \right]^{-1},$$

where $\widehat{u}_{t,\tau}^{2} = y_{t,\tau} / (\widehat{\eta}_{t} \widehat{\phi}_{\tau} \widehat{\mu}_{t,\tau} \widehat{e}_{t,\tau}^{*}) - 1$.

G.1 Estimation results

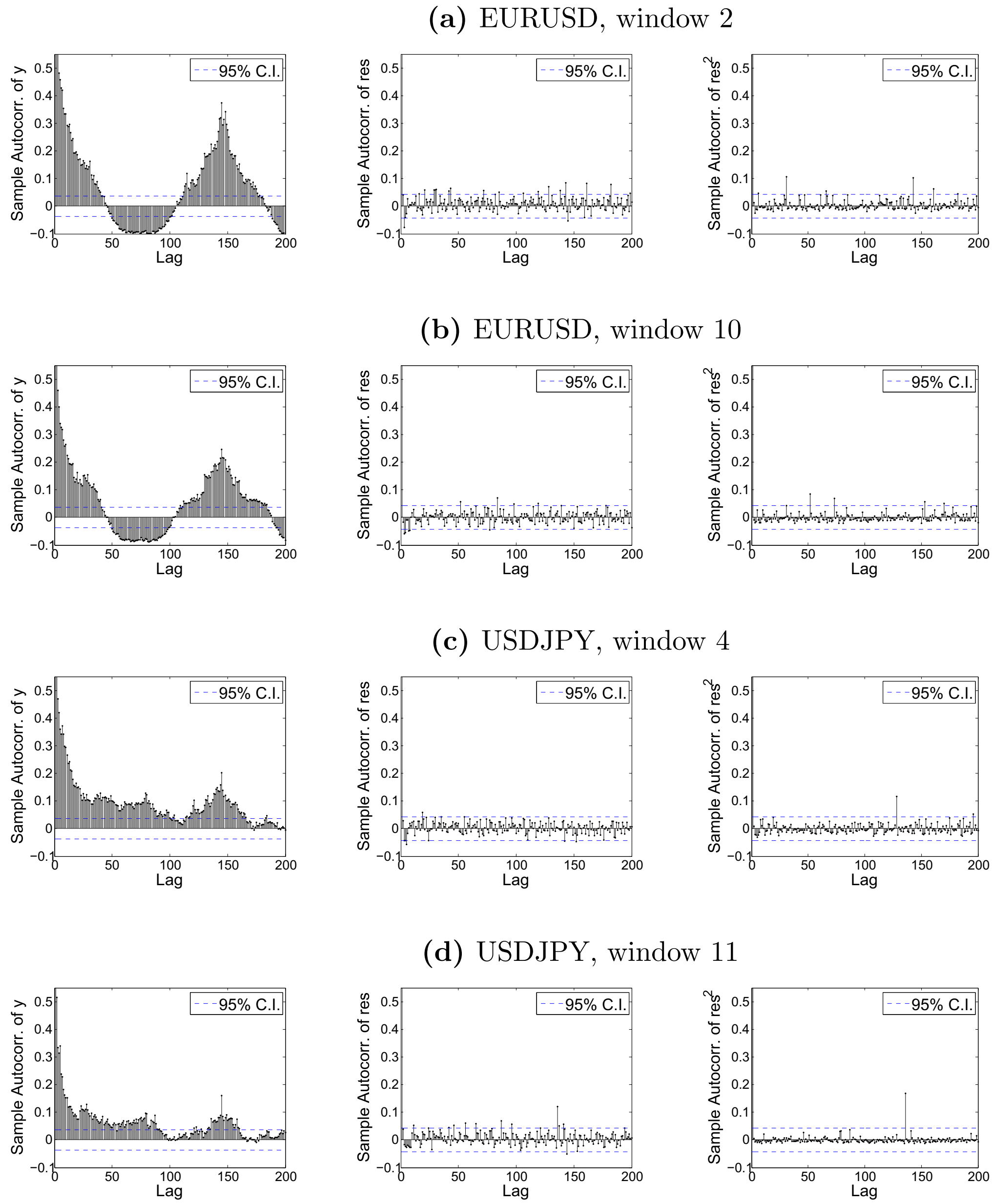

Figure 1. The sample autocorrelation of trade volume ($y_{t,\tau}$, left), $\widehat{\varepsilon}_{t,\tau}$ (middle), and $\widehat{\varepsilon}^2_{t,\tau}$ (right) in Fourier-MEM estimated by GMM. The sampling frequency is 10 minutes. The 95% confidence intervals are based on the numerical standard errors.

(a) EURUSD

	$\widehat{\varepsilon}_{t,\tau}$						$\widehat{\varepsilon}^2_{t,\tau}$					
Window	$\widehat{\rho}_1$	$\widehat{\rho}_{day}$	Q_1	Q_{day}	p-val. 1	p-val. day	$\widehat{\rho}_1$	$\widehat{\rho}_{day}$	Q_1	Q_{day}	p-val. 1	p-val. day
1	0.005	-0.005	0.060	142.769	0.807	0.000	-0.006	-0.004	0.066	11.130	0.797	1.000
2	0.038	0.021	3.186	228.207	0.074	0.000	0.012	-0.002	0.297	140.553	0.586	0.000
3	0.017	-0.035	0.610	215.108	0.435	0.000	0.001	-0.021	0.001	123.142	0.971	0.000
4	-0.004	-0.036	0.031	159.340	0.860	0.000	0.003	-0.027	0.024	147.307	0.878	0.000
5	-0.014	0.004	0.397	115.537	0.529	0.000	-0.006	-0.005	0.078	34.249	0.781	0.644
6	-0.007	0.019	0.116	149.841	0.733	0.000	-0.012	0.020	0.286	120.798	0.593	0.000
7	-0.027	0.005	1.496	144.800	0.221	0.000	-0.010	-0.005	0.196	58.338	0.658	0.024
8	0.003	0.033	0.016	144.045	0.900	0.000	0.002	0.002	0.008	27.122	0.929	0.883
9	-0.024	0.041	1.212	179.660	0.271	0.000	-0.015	0.069	0.482	119.112	0.488	0.000
10	-0.009	0.006	0.173	137.055	0.677	0.000	-0.006	-0.007	0.087	52.116	0.768	0.063
11	0.023	0.004	1.143	213.948	0.285	0.000	0.012	0.003	0.286	109.275	0.593	0.000
12	-0.003	-0.007	0.016	122.563	0.899	0.000	-0.005	-0.006	0.062	50.420	0.803	0.086
13	0.008	-0.016	0.146	160.541	0.703	0.000	0.003	-0.017	0.017	114.118	0.895	0.000
14	0.028	-0.018	1.739	162.601	0.187	0.000	0.006	0.001	0.090	103.395	0.765	0.000
15	0.049	-0.031	5.188	194.885	0.023	0.000	0.013	-0.017	0.379	167.052	0.538	0.000
16	-0.021	-0.018	0.825	132.971	0.364	0.000	-0.002	-0.002	0.007	2.656	0.935	1.000

(b) USDJPY

Window	$\widehat{\rho}_1$	$\widehat{\rho}_{day}$	Q_1	Q_{day}	p-val. 1	p-val. day	$\widehat{\rho}_1$	$\widehat{\rho}_{day}$	Q_1	Q_{day}	p-val. 1	p-val. day
1	0.040	-0.039	3.405	208.051	0.065	0.000	0.012	-0.022	0.330	95.807	0.566	0.000
2	0.054	-0.012	6.219	175.668	0.013	0.000	0.006	-0.009	0.071	91.761	0.790	0.000
3	0.034	0.016	2.548	162.420	0.110	0.000	0.008	0.018	0.140	71.724	0.708	0.001
4	0.000	0.004	0.000	129.530	0.991	0.000	0.000	0.002	0.000	78.487	0.992	0.000
5	0.023	-0.005	1.108	282.144	0.293	0.000	-0.001	-0.001	0.002	50.904	0.966	0.079
6	0.085	0.036	15.300	210.363	0.000	0.000	0.066	0.033	9.243	161.357	0.002	0.000
7	0.003	0.035	0.021	56.842	0.884	0.032	-0.002	0.004	0.007	0.678	0.935	1.000
8	0.019	0.009	0.790	151.175	0.374	0.000	0.000	0.011	0.000	29.212	0.984	0.846
9	0.039	0.028	3.260	189.737	0.071	0.000	0.009	0.048	0.171	247.381	0.680	0.000
10	0.051	0.048	5.558	213.444	0.018	0.000	0.002	0.053	0.012	130.015	0.914	0.000
11	0.030	0.003	1.929	210.740	0.165	0.000	0.002	-0.004	0.006	84.452	0.939	0.000
12	0.038	0.001	3.076	152.323	0.079	0.000	-0.001	-0.004	0.005	164.788	0.945	0.000
13	0.046	0.039	4.467	103.816	0.035	0.000	0.000	0.010	0.000	2.641	0.985	1.000
14	0.026	-0.042	1.423	169.537	0.233	0.000	0.007	-0.032	0.110	146.309	0.740	0.000
15	0.162	0.044	56.645	175.068	0.000	0.000	0.038	0.001	3.050	4.265	0.081	1.000
16	0.005	0.017	0.044	26.517	0.833	0.848	-0.001	0.000	0.003	0.351	0.957	1.000

Table 3. Residual analysis for Fourier-MEM. The sampling frequency is 10 minutes. Q_l is the Ljung-Box statistic to test the null of no autocorrelation up to the l-th lag.

Pair	EURUSD							
Freq	10mins							
Window	3		6		9		12	
	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error
$\alpha_0^{(\eta)}$	4.239	1.707	3.064	1.577	2.981	1.343	12.726	2.649
$\beta_1^{(\eta)}$	0.504	0.016	0.410	0.019	0.486	0.017	0.516	0.018
$\alpha_1^{(\eta)}$	0.495	0.017	0.497	0.019	0.478	0.017	0.420	0.018
$\beta_1^{(e)}$	0.699	0.105	0.500	0.159	0.628	0.206	0.681	0.137
$\beta_1^{(\mu)}$	0.498	0.005	0.507	0.002	0.517	0.004	0.534	0.002
$\alpha_1^{(\mu)}$	0.336	0.004	0.353	0.002	0.322	0.003	0.358	0.002
$\alpha_2^{(\mu)}$	0.000	0.007	0.018	0.004	0.000	0.006	0.097	0.004
$\alpha_{1,1}^{(e)}$	1.291	0.457	0.936	0.521	0.221	0.123	2.110	0.530
$\alpha_{1,2}^{(e)}$	0.569	0.132	0.682	0.132	-0.105	0.874	0.302	0.137
$\alpha_{1,3}^{(e)}$	-0.609	0.864	0.582	1.001	0.792	0.920	-0.135	0.970

Pair	USDJPY							
Freq	10mins							
Window	1		6		8		10	
	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error
$\alpha_0^{(\eta)}$	6.335	3.006	3.162	1.672	4.709	2.539	13.490	2.016
$\beta_1^{(\eta)}$	0.525	0.016	0.512	0.018	0.520	0.019	0.431	0.021
$\alpha_1^{(\eta)}$	0.498	0.017	0.505	0.019	0.503	0.020	0.516	0.023
$\beta_1^{(e)}$	0.844	0.083	0.616	0.155	0.705	0.179	0.800	0.060
$\beta_1^{(\mu)}$	0.496	0.003	0.512	0.002	0.483	0.001	0.526	0.003
$\alpha_1^{(\mu)}$	0.363	0.002	0.423	0.002	0.389	0.001	0.373	0.002
$\alpha_2^{(\mu)}$	0.075	0.005	0.005	0.004	0.041	0.002	0.000	0.004
$\alpha_{1,1}^{(e)}$	2.171	0.730	1.793	0.524	-2.015	0.865	3.763	0.709
$\alpha_{1,2}^{(e)}$	0.542	0.193	0.510	0.187	1.064	0.675	1.313	0.671
$\alpha_{1,3}^{(e)}$	0.730	0.929	0.765	0.980	0.567	0.214	0.258	0.162

Table 4. The estimated parameter values for Fourier-MEM. The results are shown for selected sampling windows. The standard errors are computed analytically using the asymptotic results outlined above. The parameters of ϕ_τ are excluded.

References

Brownlees, C. T., Cipollini, F., and Gallo, G. M. (2011), "Intra-Daily Volume Modelling and Prediction for Algorithmic Trading," *Journal of Financial Econometrics*, 9, 489–518.

Harvey, A. C. and Koopman, S. J. (1993), "Forecasting Hourly Electricity Demand Using Time-Varying Splines," *Journal of the American Statistical Association*, 88, 1228–1236.

Harvey, A. C. and Lange, R.-J. (2015), "Volatility Modeling with a Generalized t-Distribution," Cambridge Working Papers in Economics CWPE1517, University of Cambridge.

Kleiber, C. and Kotz, S. (2003), *Statistical Size Distributions in Economics and Actuarial Sciences*, Wiley: New York.

Poirier, D. (1976), *The Econometrics of Structural Change: With Special Emphasis on Spline Functions*, North-Holland.