# Solutions for Linear Algebra
# Preparatory course 2014

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## Contents

**Exercise 1.** Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{z}$ be vectors of the same order. Show that

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})+\mathbf{z}=\mathbf{v}+(\mathbf{u}+\mathbf{z})$

**Solution 1.** It suffices to show that corresponding components of each side are equal:

(a) $u_i+v_i=v_i+u_i$

(b) $(u_i+v_i)+z_i=v_i+(u_i+z_i)$

**Exercise 2.** For vectors $\mathbf{u}$ and $\mathbf{v}$ of the same order and scalars $c$ and $d$. Show that

(a) $(c+d)(\mathbf{u}+\mathbf{v})=c\mathbf{v}+c\mathbf{u}+d\mathbf{v}+d\mathbf{u}$

(b) The zero vector is uniquely determined by the condition that $c\mathbf{0}=\mathbf{0}$ for all finite scalars $c$.

**Solution 2.** (a) $i^{th}$ componenent of LHS is $(c+d)(v_i+u_i)$, $i^{th}$ component of RHS is $cv_i+cu_i+dv_i+du_i$. The two sides are equal by scalar arithmetic.

(b) The statement is that $c\mathbf{v}=\mathbf{v}\ \forall c<\infty \Rightarrow \mathbf{v}=\mathbf{0}$, where $c\mathbf{v}=\mathbf{v}$ is equivalent to $(c-1)v_i=0\ \forall i$.

Suppose not. Suppose $\exists i$ s.t. $v_i\neq 0$, then $(c-1)v_i\neq 0$ except for $c=1$. This contradicts the statement $c\mathbf{v}=\mathbf{v}$.

**Exercise 3.** Show that $(A+B)^T = A^T + B^T$ for any two conformable matrices $A$ and $B$.

**Solution 3.** $(A+B)^T = (a_{ji} + b_{ji}) = (a_{ji}) + (b_{ji}) = A^T + B^T$.

**Exercise 4.** Write the following system in matrix form and draw the corresponding *row picture* and *column picture*. By simple inspection, identify the $\mathbf{x}$ vector that solves the system.

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

**Solution 4.**

**Exercise 5.** Write the following system in matrix form and draw the corresponding *row picture* and *column picture*. By simple inspection, identify the $\mathbf{x}$ vector that solves the system.

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned}$$

**Solution 5.**

**Exercise 6.** Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be linearly independent vectors of order $n$, show that $(\mathbf{x}+\mathbf{y})$, $(\mathbf{x}+\mathbf{z})$, and $(\mathbf{y}+\mathbf{z})$ are also linearly independent.

**Solution 6.** $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly independent, i.e. there exists no set of scalars $c_1, c_2, c_3$ (at least one of which non-zero) such that $c_1\mathbf{z} + c_2\mathbf{x} + c_3\mathbf{y} = \mathbf{0}$.

Claim: $(\mathbf{x}+\mathbf{y})$, $(\mathbf{x}+\mathbf{z})$, and $(\mathbf{y}+\mathbf{z})$ are linearly independent.

Suppose not. Then there exists constants $d_1$, $d_2$, $d_3$ – at least one of which non-zero – such that

$$\mathbf{0} = d_1(\mathbf{x}+\mathbf{y}) + d_2(\mathbf{x}+\mathbf{z}) + d_3(\mathbf{y}+\mathbf{z})$$

i.e.

$$\mathbf{0} = (d_1 + d_2)\mathbf{x} + (d_1 + d_3)y + (d_2 + d_3)\mathbf{z} \tag{0.1}$$

but $d_1$, $d_2$ and $d_3$ are not all zero, and $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly independent, a contradiction.

**Exercise 7.** Let $A$ be an $n \times p$ matrix. Suppose you know that there exists an $n \times 1$ vector $\mathbf{a}$ such that if $\mathbf{a}$ is added as an additional column to $A$, the rank of increases by 1. i.e. $r(A : \mathbf{a}) = r(A) + 1$. Show that this implies the rows of $A$ are linearly dependent. **Hint:** prove by contradition.

**Solution 7.** Suppose not. Suppose $r(A : a) = r(A) + 1$ and rows are linearly independent, i.e. $r(A) = n$. First consider the case in which $n > p$. $r(A) = \min(n, p) = p < n$. A contradiction. Now consider the case in which $n < p$ or $n = p$. Since $r(A : a) \leq \min(n, p + 1) = n$, $r(A : a) = r(A)$. A contradiction.

**Exercise 8.** Show that it is not true in general that $r(AB) = r(BA)$ for two square matrices $A$ and $B$.

**Solution 8.** It suffices to provide a counterexample. Let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

then

$$r(AB) = r\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{and} \quad r(BA) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1$$

**Exercise 9.** $A$ is $3 \times 5$, $B$ is $5 \times 3$, $C$ is $5 \times 1$ and $D$ is $3 \times 1$. Which of these matrix operations are allowed?

$$BA \qquad AB \qquad ABC \qquad DBA \qquad A(B + C)$$

**Solution 9.** Only $BA$ and $AB$ are allowed.

**Exercise 10.** Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

Compute $AB^T$, $BA^T$, $A^TB$, and $B^TA$

**Solution 10.**

$$AB^T = \begin{pmatrix} 32 & 14 \\ 77 & 32 \end{pmatrix} \qquad BA^T = \begin{pmatrix} 32 & 77 \\ 14 & 32 \end{pmatrix}$$

$$A^TB = \begin{pmatrix} 8 & 13 & 18 \\ 13 & 20 & 27 \\ 18 & 27 & 36 \end{pmatrix} \qquad B^TA = \begin{pmatrix} 8 & 13 & 18 \\ 13 & 20 & 27 \\ 18 & 27 & 36 \end{pmatrix}$$

**Exercise 11.** .

(a) Show that $(AB)^T = B^TA^T$

(b) Show that $(ABC)^T = C^TB^TA^T$. Hint: let $D := BC$ and use (a).

(c) Under what condition is $(AB)^T = A^TB^T$?

(d) Find an example of two matrices $A$ and $B$ such that $AB = BA$.

**Solution 11.** (a)

$$\begin{aligned}(B^TA^T)_{ij} &= \sum_k (B^T)_{ik}(A^T)_{kj} \\ &= \sum_k (B)_{ki}(A)_{jk} \\ &= \sum_k (A)_{jk}(B)_{ki} = (AB)_{ji}\end{aligned}$$

(b) Let $D = BC$

$$(ABC)^T = (AD)^T = D^TA^T[\text{by part (a)}] = (BC)^TA^T = C^TB^TA^T[\text{by part (a) again}]$$

(c) $(AB)^T = B^TA^T$ $B^TA^T = A^TB^T$ when $AB = BA$, i.e. when matrices commute.

(d) Let $A$ be the identity matrix. Then whatever is $B$, $AB = BA$.

**Exercise 12.** For square matrices $A$ and $B$, which of the following statements are true, and why?

(a) $(A + B)^2 = (B + A)^2$

(b) $(A + B)^2 = A^2 + 2AB + B^2$

(c) $(A + B)^2 = A(A + B) + B(A + B)$

(d) $(A + B)^2 = (A + B)(B + A)$

(e) $(A + B)^2 = A^2 + AB + BA + B^2$

**Solution 12.** (a) True. $A + B = B + A$

(b) False. $AB \neq BA$ in general

(c) True. $A(A + B) + B(A + B) = (A + B)(A + B)$.

(d) True.

(e) True.

**Exercise 13.** For square matrices $A$ and $B$, which of the following statements are true, and why?

(a) $(A - B)^2 = (B - A)^2$

(b) $(A - B)^2 = A^2 - B^2$

(c) $(A - B)^2 = A^2 - 2AB + B^2$

(d) $(A-B)^2 = A(A-B) - B(A-B)$

(e) $(A-B)^2 = A^2 - AB - BA + B^2$

**Solution 13.** (a) True.

(b) False.

(c) False.

(d) True.

(e) True.

**Exercise 14.** What rows or columns or matrices do you multiply to find

(a) the third column of $AB$?

(b) the first row of $AB$?

(c) the entry in row three, column 4 of $AB$?

(d) the entry in row 1, column 1 of $CDE$?

**Solution 14.** (a) (Use *matrix times column*) Matrix $A$ times column 3 of $B$.

(b) (Use *row times matrix*) First row of $A$ times matrix $B$.

(c) Row 3 of $A$ times columns 4 of $B$.

(d) (Use *row times matrix and matrix times columns*) Row ones of $C$ times $D$ times column 1 of $E$.

**Exercise 15.** An econometrician collects data on the number of years of education and the marital status of $n$ different individuals, i.e. for the $i^{th}$ individual he has a $2 \times 1$ vector $\mathbf{x}_i$ and he arranges these $n$ 2-dimensional observations into a matrix $X$ of order $n \times 2$. Show that $X^T X = \sum_i \mathbf{x}_i \mathbf{x}_i^T$

**Solution 15.**

$$\underbrace{\mathbf{x}_i}_{(2\times 1)} = \begin{pmatrix} | \end{pmatrix} \qquad \underbrace{X}_{(n\times 2)} = \begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_n^T \end{pmatrix} = \begin{pmatrix} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{pmatrix}$$

$$\underbrace{X^T X}_{(2\times 2)} = \begin{pmatrix} | & | & \dots & | \end{pmatrix} \begin{pmatrix} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{pmatrix} = \sum_i \mathbf{x}_i \mathbf{x}_i^T$$

**Exercise 16.** The econometrician also collects data on income of the same $n$ individuals. Let $y$ be the $n \times 1$ vector with $i^{th}$ element $y_i$. Show that $X^T y = \sum_i \mathbf{x}_i y_i$

**Solution 16.**

$$\underbrace{X^T y}_{(2\times 1)} = \begin{pmatrix} | & | & \dots & | \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_i \mathbf{x}_i y_i$$

**Exercise 17.** Suppose that for every $i = 1, \dots, n$, the econometrician applies a weight $1/\sigma_i$ to the $i^{th}$ observation and arranges these $n$ 2-dimensional observations into a matrix $Z$ of order $n \times 2$ such that the $ij^{th}$ element of $Z$ is $x_{ij}/\sigma_i$. Let $\Omega$ be the diagonal matrix with $ii^{th}$ element $\sigma_i \sigma_i = \sigma_i^2$. Show that

$$Z^T Z = \sum_i (\mathbf{x}_i/\sigma_i)(\mathbf{x}_i/\sigma_i)^T = \sum_i \mathbf{x}_i \sigma_i^{-2} \mathbf{x}_i^T = X^T \Omega^{-1} X$$

and

$$\sum_i (\mathbf{x}_i/\sigma_i) y_i/\sigma_i = \sum_i \mathbf{x}_i \sigma_i^{-2} y_i = X^T \Omega^{-1} \mathbf{y}$$

**Solution 17.**

$$\underbrace{Z}_{(n\times 2)} = \begin{pmatrix} \frac{1}{\sigma_1}\mathbf{x}_1^T \\ \frac{1}{\sigma_2}\mathbf{x}_2^T \\ \vdots \\ \frac{1}{\sigma_n}\mathbf{x}_n^T \end{pmatrix} \qquad \Omega = \begin{pmatrix} \sigma_1^2 & 0 & \cdot & \cdot & 0 \\ 0 & \sigma_2^2 & 0 & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \sigma_n^2 \end{pmatrix} \qquad \Omega^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & \cdot & \cdot & 0 \\ 0 & \frac{1}{\sigma_2^2} & 0 & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \frac{1}{\sigma_n^2} \end{pmatrix}$$

$$Z^T Z = \sum_i \frac{1}{\sigma_i}\mathbf{x}_i \frac{1}{\sigma_i}\mathbf{x}_i^T = \sum_i \frac{1}{\sigma_i^2}\mathbf{x}_i \mathbf{x}_i^T$$

$$X^T \Omega^{-1} X = \underbrace{\begin{pmatrix} | & | & \dots & | \end{pmatrix}}_{(2\times n)} \underbrace{\underbrace{\begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & \cdot & \cdot & 0 \\ 0 & \frac{1}{\sigma_2^2} & 0 & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \frac{1}{\sigma_n^2} \end{pmatrix}}_{(n\times n)} \underbrace{\begin{pmatrix} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{pmatrix}}_{(n\times 2)}}_{\sum_i \mathbf{\Omega}_i^{-1} \mathbf{x}_i^T}$$

Where

$$\sum_i \boldsymbol{\Omega}_i^{-1}\mathbf{x}_i^T = \begin{pmatrix} \frac{x_{11}}{\sigma_1^2} & \frac{x_{12}}{\sigma_1^2} \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{x_{21}}{\sigma_2^2} & \frac{x_{22}}{\sigma_2^2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \frac{x_{n1}}{\sigma_\text{n}^2} & \frac{x_{n2}}{\sigma_\text{n}^2} \end{pmatrix} = \begin{pmatrix} \frac{x_{11}}{\sigma_1^2} & \frac{x_{12}}{\sigma_1^2} \\ \frac{x_{21}}{\sigma_2^2} & \frac{x_{22}}{\sigma_2^2} \\ \vdots & \vdots \\ \frac{x_{n1}}{\sigma_\text{n}^2} & \frac{x_{n2}}{\sigma_\text{n}^2} \end{pmatrix}$$

So

$$X^T\Omega^{-1}X = \underbrace{\begin{pmatrix} | & | & \ldots & | \end{pmatrix}}_{(2\times n)} \underbrace{\begin{pmatrix} \frac{x_{11}}{\sigma_1^2} & \frac{x_{12}}{\sigma_1^2} \\ \frac{x_{21}}{\sigma_2^2} & \frac{x_{22}}{\sigma_2^2} \\ \vdots & \vdots \\ \frac{x_{n1}}{\sigma_\text{n}^2} & \frac{x_{n2}}{\sigma_\text{n}^2} \end{pmatrix}}_{(n\times 2)} = \sum_i \mathbf{x}_i\mathbf{x}_i^T/\sigma_i^2 = Z^TZ$$

**Exercise 18.** Multiply $A$ (order $3 \times 3$) and $I_3$ using columns of $A$ times rows of $I_3$.

**Solution 18.**

$$\begin{aligned} AI_3 &= \begin{pmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} | \\ \mathbf{a}_2 \\ | \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} | \\ \mathbf{a}_3 \\ | \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A \end{aligned}$$

**Exercise 19.** Multiply $AB$ using *columns times rows*:

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \end{pmatrix} + \underline{\qquad\qquad} = \underline{\qquad\qquad}$$

**Solution 19.**

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{pmatrix}$$

**Exercise 20.** Construct all the permutation matrices of order $p = 3$. How many permutation matrices of order $p = 4$ are there?

**Solution 20.**

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad P_6 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For a $p \times p$ identity matrix there are $p!$ permutations. So there are 24 permutation matrices of order 4.

**Exercise 21.** Suppose you are given 50 points on the surface of a sphere. What is the dimension of the matrix you would require in order to map those points to 50 points on the surface of a 3-dimensional ellipsoid?

**Solution 21.** You need a matrix that will map from a three dimensional space to a three dimensional space.

$$\underbrace{\begin{pmatrix} & (3 \times 3) & \end{pmatrix}}_{\text{[transformation matrix]}} \underbrace{\begin{pmatrix} & (3 \times 50) & \end{pmatrix}}_{\text{[50 points on the sphere]}} = \underbrace{\begin{pmatrix} & (3 \times 50) & \end{pmatrix}}_{\text{[50 points on the ellipsoid]}}$$

**Exercise 22.** You are given the matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \\ x_{51} & x_{52} & x_{53} \end{pmatrix}$$

Construct a matrix $A$ such that

$$AX = \begin{pmatrix} x_{21} - x_{11} & x_{22} - x_{12} & x_{23} - x_{13} \\ x_{31} - x_{21} & x_{32} - x_{22} & x_{33} - x_{23} \\ x_{41} - x_{31} & x_{42} - x_{32} & x_{43} - x_{33} \\ x_{51} - x_{41} & x_{52} - x_{42} & x_{53} - x_{43} \end{pmatrix}$$

**Solution 22.**

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

**Exercise 23.** Prove property (4) using property (2). Hint: force a contradition by supposing $A$ had two equal rows and $det(A) = a \neq 0$.

**Solution 23.** Suppose not. Suppose $A$ has 2 equal rows and $\det(A) = a \neq 0$. Then by property (2) we can exchange the two rows of $A$ producing $\bar{A}$ with $\det(A) = -a \neq 0$. But $A = \bar{A}$ so $\det(A) = -a$, a contradiction.

**Exercise 24.** Prove property (5) using properties (3) and (4). Hint: Use both parts of property (3) to separate the relevent determinant into two components and apply property (4).

**Solution 24.**

$$\begin{vmatrix} a & b \\ c - la & lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{=0 \text{ [property (4)]}}$$

**Exercise 25.** Prove property (6) using property (3).

**Solution 25.** Use property 3(a) with $t = 0$.

**Exercise 26.** By considering the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$$

and the properties of determinants, show that $AB = 0$ does not imply that either $A$ or $B$ is the zero matrix but that it does imply that at least one of them is singular.

**Solution 26.** Notice that $AB$ is equal to the zero matrix despite neither $A$ or $B$ being equal to the zero matrix. By the properties of determinants we have $0 = \det(AB) = \det(A)\det(B)$, therefore either $\det(A) = 0$ or $\det(B) = 0$.

**Exercise 27.** Use properties (1), (2), (3), (6) and (8) above to show that

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

**Solution 27.** By property (3), write

$$\begin{aligned}\begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\ &= \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{=0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{=0} \\ &= ad\underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{=1} + bc\underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{=-1} \\ &= ad - bc\end{aligned}$$

**Exercise 28.** Use the same method (hence properties (1), (2), (3), (6) and (8) above) to show that

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}$$

**Solution 28.** Write

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \\ &+ \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{31}a_{22}a_{13} + a_{13}a_{21}a_{32}\end{aligned}$$

**Exercise 29.** Prove theorem 1:

**Theorem 1.** *If an inverse exists, then it is unique.*

**Hint:** force a contradiction by introducing matrices $B$ and $C \neq B$ that both satisfy the definition of the inverse, i.e.

$$AB = BA = I \qquad \text{and} \qquad AC = CA = I$$

**Solution 29.** Suppose not. Suppose an inverse exists and is not unique, i.e. there exist two square matrices $B$ and $C \neq B$ such that

$$AB = BA = I \quad \text{and} \quad AC = CA = I$$

Then

$$B = B(AC) = (BA)C = C$$

A contradiction to the claim that $B \neq C$.

**Exercise 30.** Use theorem 1 to prove theorem 2:

**Theorem 2.** *If $A$ is a square matrix of order $p$ with an inverse, then the system of equations*

$$\sum_{j=1}^{p} a_{ij}x_j = b_i \qquad (1 \leq i \leq p)$$

*has a unique solution for each choice of $b_i$*

**Solution 30.** For existence, set $\mathbf{x} = A^{-1}\mathbf{b}$, then

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}$$

so

$$\sum_{j=1}^{p} a_{ij}x_j = b_i \ \ \forall\ (1 \leq i \leq p)$$

For uniqueness, if $\sum_{j=1}^{p} a_{ij}x_j = b_i \ \ \forall\ (1 \leq i \leq p)$ then $A\mathbf{x} = \mathbf{b}$ and

$$\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

The solution is unique by the uniqueness of the inverse.

**Exercise 31.** Construct a matrix $A$ that multiplies the vector $(3,\ -1)^T$ to produce the zero vector $(0,\ 0)^T$. What do you notice about the matrix $A$? Compute its determinant.

**Solution 31.**

$$A \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

One such matrix $A$ is

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

Notice that the rows and columns are multiples of one another. $\det(A) = 6 - 6 = 0$

**Exercise 32.** Consider the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 7 \\ 3 & 6 & 2 \end{pmatrix}$$

and take note of its rank. Show, using theorem **??** and the properties of determinants that the matrix in the above display is non-invertible.

**Solution 32.**

$$r\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 7 \\ 3 & 6 & 2 \end{pmatrix} = 2$$

($2^{nd}$ column is two times the $1^{st}$)

$$\underbrace{\begin{vmatrix} 1 & 2 & 0 \\ 2 & 4 & 7 \\ 3 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 7 & 2 \end{vmatrix}}_{by property (8)}$$

$$\underbrace{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 7 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 0 & 7 & 2 \end{vmatrix}}_{by property (2)} = 0 \text{ by property (4)}$$

**Exercise 33.** Suppose $A$ and $B$ are matrices of the same order and both have an inverse. Show that

$$(AB)^{-1} = B^{-1}A^{-1}$$

**Solution 33.** By definition, the inverse of $AB$ satisfies

$$AB(\text{inverse of } AB) = (\text{inverse of } AB)AB = I$$

$$AB(B^{-1}A^{-1}) = AIA^{-1} = I$$

and

$$(B^{-1}A^{-1})AB = B^{-1}IB = I$$

As required.

**Exercise 34.** For any nonsingular square matrix $A$, show that

(a) $(cA)^{-1} = (1/c)A^{-1}$ $(c \neq 0)$

(b) $(A^{-1})^T = (A^T)^{-1}$

(c) $(A^{-1})^{-1} = A$

**Solution 34.** (a) $(cA)(1/c)A^{-1} = I$ and $(1/c)A^{-1}cA = I$

(b) $A^T(A^{-1})^T = (A^{-1}A)^T = I$ and $(A^{-1})^T A^T = (AA^{-1})^T = I$

(c) If $(A^{-1})^{-1} = A$, then $(A^{-1})^{-1}$ satisfies

$$A^{-1}(A^{-1})^{-1} = (A^{-1})^{-1}A^{-1} = I$$

$$A^{-1}(A^{-1})^{-1} = (A^{-1}A)^{-1} = I$$

and

$$(A^{-1})^{-1}A^{-1} = (AA^{-1})^{-1} = I$$

as required.

**Exercise 35.** Let $A$ and $B$ be square matrices of the same order and let $C := AB$ be invertible. This means that $A$ is invertible. Find an expression for $A^{-1}$ in terms of $C^{-1}$ and $B$.

**Solution 35.** Postmultiplying both sides of $C = AB$ by $C^{-1}$ gives

$$I = ABC^{-1}$$

Premultiplying both sides by $A^{-1}$ gives

$$A^{-1} = BC^{-1}$$

**Exercise 36.** Show that a matrix with a column of zeros cannot have an inverse.

**Solution 36.** Just apply properties (8) and (6) of determinants to show the determinant is zero, then apply Theorem 4 from the notes.

**Exercise 37.** Suppose $A$ is invertible and you exchange its first two rows to obtain $B$. Explain why the new matrix is invertible. How would you find $B^{-1}$ from $A^{-1}$.

**Solution 37.** Let $B$ be the matrix obtained by exchanging the first two rows of $A$. Then by property (2) of determinants, $\det(A) = -\det(B)$. Since $A$ is invertible $\det(A) \neq 0$ so $\det(B) \neq 0$. B is invertible.

Notice that $B = PA$ where

$$\begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & 0 & & & & \cdot \\ 0 & 0 & 1 & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & 0 \\ 0 & \cdot & \cdot & & & 0 & 1 \end{pmatrix}$$

$$PA = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} \begin{pmatrix} a_{21} & a_{22} & \dots & a_{2n} \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

So interchanging the first and second rows leaves the rest of the matrix unchanged.

$$B^{-1} = (PA)^{-1} = A^{-1}P^{-1} = A^{-1}P$$

where the last equality follows by the special inverse rules in section 5.2.2 of the notes. i.e. the inverse of $B$ is obtained by interchaning the first two columns of the inverse of $A$.

**Exercise 38.** Refering to your computations in exercise 10 above,

(a) Check that

$$tr(A^T B) = tr(BA^T) = tr(AB^T) = tr(B^T A)$$

(b) Prove that properties (1) and (2) of the trace hold for arbitrary $n \times p$ matrices $A$ and $B$, hence justify the above display for such matrices.

**Solution 38.** (a) Direct computation gives $tr(A^T B) = tr(BA^T) = tr(AB^T) = tr(B^T A) = 64$

(b) Since $(A^T B)^T = B^T A$, $(BA^T)^T = AB^T$ and $tr(A^T) = tr(A)$, it suffices to prove that $tr(A^T B) = tr(BA^T)$ (this is the missing link).

$$\begin{aligned} tr(A^T B) &= \sum_j (A^T B)_{jj} \\ &= \sum_j \left( \sum_k a_{kj} b_{kj} \right) \\ &= \sum_k \left( \sum_j b_{kj} a_{kj} \right) = \sum_k (BA^T)_{kk} =: tr(BA^T) \end{aligned}$$

**Exercise 39.** Find the two eigenvalues of the matrix

$$B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

by finding the characteristic roots of an equation of the form

$$\alpha\lambda^2 + \beta\lambda + \gamma = 0$$

Deduce two eigenvectors of $B$ by inspection.

**Solution 39.**

$$\det(B - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3=\lambda \end{vmatrix} = (3-\lambda)^2 - 1$$

Solving the characteristic equation $\det(B - \lambda I) = 0$ for $\lambda$ gives

$$(\lambda - 4)(\lambda - 2) = 0 \qquad \Rightarrow \lambda_1 = 4, \lambda_2 = 2$$

Two possible eigenvectors that solve the eigenequations $A\mathbf{x}_1 = 4\mathbf{x}_1$ and $A\mathbf{x}_2 = 2\mathbf{x}_2$ are

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Notices that there are infinitely many eigenvectors that satisfy each of the two eigenequations, and they all lie on the same line, in practice we choose the most natural, or the ones that satisfy $\mathbf{x}^T\mathbf{x} = 1$.

**Exercise 40.** Let $A$ be the permutation matrix in example **??**, and write the matrix $B$ of exercise 39 as $B = (A + 3I)$. Algebraically, deduce that the eigenvalues of $A$ are three less than the eigenvalues of $B$ and that the eigenvectors are unchanged.

**Solution 40.**

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A + 3I$$

The eigenvectors of $B$ are defined as the vectors $\mathbf{x}$ that satisfy $B\mathbf{x} = \lambda\mathbf{x}$ i.e. $(A + 3I)\mathbf{x} = \lambda\mathbf{x}$. Multiplying out the bracket, $A\mathbf{x} + 3\mathbf{x} = \lambda\mathbf{x}$ so $A\mathbf{x} = (\lambda - 3)\mathbf{x}$, as required.

**Exercise 41.** Using $A\mathbf{x} = \lambda\mathbf{x}$, show that

(a) $\lambda^2$ is an eigenvalue of $A^2$

(b) $\lambda^{-1}$ is an eigenvalue of $A^{-1}$

(c) $\lambda + 1$ is an eigenvalue of $A + I$.

**Solution 41.** (a) Since $\lambda$ is an eigenvalue of $A$, $A\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x}$ is an eigenvector associated with $\lambda$. Premultiplying by $A$ gives

$$A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$$

(b) Similarly

$$A^{-1}A\mathbf{x} = \lambda A^{-1}\mathbf{x}$$

so

$$(1/\lambda)\mathbf{x} = A^{-1}\mathbf{x}$$

We have recovered the eigenequation for $A^{-1}$, so $1/\lambda$ is an eigenvalue of $A^{-1}$.

(c)

$$(A + I)\mathbf{x} = A\mathbf{x} + I\mathbf{x} = \lambda\mathbf{x} + I\mathbf{x} = (\lambda + 1)\mathbf{x}$$

**Exercise 42.** Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(a) Show that $\det(A + \lambda I) = \lambda^2 - tr(A)\lambda + \det(A)$.

(b) Find an expression for the two eigenvalues of $A$ in terms of $a$, $b$, $c$ and $d$. These characteristic roots satisfy

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - tr(A)\lambda + \det(A) = 0$$

and $\lambda_1\lambda_2 = \det(A)$ and $\lambda_1 + \lambda_2 = tr(A)$ (feel free to check that $\lambda_1\lambda_2$ does indeed match $\det(A) = ad - bc$ and $\lambda_1 + \lambda_2$ matches $tr(A) = (a + d)$)

**Solution 42.** (a)

$$\det(A + \lambda I) = (a - \lambda)(d - \lambda) - bc = ad - bc - a\lambda - d\lambda + \lambda^2 = \lambda^2 - \underbrace{(a + d)}_{tr(A)}\lambda + \underbrace{(ad - bc)}_{\det A}$$

(b)

$$\lambda_1, \lambda_2 = \frac{-(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

**Exercise 43.** Assuming you can write the initialisation vector $\mathbf{u}_0$ as a linear conbination of the eigenvectors, find an expression for the $k^{th}$ element of an arbitrary sequence of the form $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

**Solution 43.** Since the initialisation vector can be written as a linear combination of eigenvectors we have

$$\mathbf{u}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_p\mathbf{x}_p = S\mathbf{c}$$

so $\mathbf{c} = S^{-1}\mathbf{u}_0$. We have

$$\begin{aligned}\mathbf{u}_k &= A^k\mathbf{u}_0 \\ &= S\Lambda^k S^{-1}\mathbf{u}_0 \\ &= S\Lambda^k S^{-1}S\mathbf{c} \\ &= c_1\lambda_1^k\mathbf{x}_1 + \cdots + c_p\lambda_p^k\mathbf{x}_p\end{aligned}$$

**Exercise 44.** Using the expression you found in exercise 43, find the $100^{th}$ element of the Fibonacci sequence in worked example (3) from the notes.

**Solution 44.** For the Fibonacci sequence,

$$\mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and as deduced previously, $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$. By construction, the only solutions to

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2}x + y \\ x - \frac{1+\sqrt{5}}{2}y \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{2}x + y \\ x - \frac{1-\sqrt{5}}{2}y \end{pmatrix}$$

are $x = y = 0$, but we can find eigenvectors by choosing $y$ and solving for $x$, giving eigenpairs

$$\lambda_1 = (1+\sqrt{5})/2\ \mathbf{x}_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = (1-\sqrt{5})/2,\ \mathbf{x}_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

To find the $c_1$ and $c_2$ such that

$$\mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} + c_2\begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

we invert the matrix $S$ whose columns are the independent eigenvectors $\mathbf{x_1}$ and $\mathbf{x}_2$.

$$S = ((\begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$$

We have

$$\mathbf{c} = S^{-1}\mathbf{u}_0 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}}(\mathbf{x_1} - \mathbf{x}_2)$$

$F_k$ is the second component of $\mathbf{u}_k$, so we have

$$F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

**Exercise 45.** Show that a positive (semi) definite matrix $A$ has positive (non-negative) eigenvalues.

**Solution 45.** Let $\lambda$ be an eigenvalue of $A$ with associated eigenvector $\mathbf{x}$, then $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda\mathbf{x}^T\mathbf{x}$. Since $A$ is positive (semi) definite $\mathbf{x}^T A\mathbf{x} > 0$ $\mathbf{x}^T A\mathbf{x} \geq 0$ and since $\mathbf{x} \neq 0$ ( $\mathbf{x}$ is an eigenvector) $\mathbf{x}^T\mathbf{x} > 0$, which provides the result.

**Exercise 46.** Show that if a matrix $A$ is symmetric and has positive (non-negative) eigenvalues, then it must be postive (semi) definite.

**Solution 46.** Using the eigendecomposition for $A$ we have

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T S\Lambda S^{-1}\mathbf{x}$$

and since $A$ is symmetric $S^{-1} = S^T$ hence letting $\mathbf{y} = S^T\mathbf{x}$, we have

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T S\Lambda S^T\mathbf{x} = \sum_{j=1}^{p} \lambda_j y_j^2 > 0$$

**Exercise 47.** Let $\Omega$ be a symmetric positive (semi) definite matrix. Use the spectral decomposition to show that there exists a nonsingular matrix $K$ such that $\Omega = K^T K$ and such that $K^{-1} = (K^T)^{-1}$. What is $K^{-1}$?

**Solution 47.** Notice that since the eigenvalues of a positive definite matrix are positive, $K = Q\Lambda^{1/2}Q^T$ satisfies the above requirements, and $K^{-1} = Q\Lambda^{-1/2}Q^T = (K^T)^{-1}$.